A CRITICAL CENTRE-STABLE MANIFOLD FOR THE SCHRÖDINGER EQUATION IN THREE DIMENSIONS

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Abstract. Consider the $\dot H^{1/2}$-critical Schrödinger equation with a focusing cubic nonlinearity in $\mathbb{R}^3$

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi = 0. \tag{0.1}$$

It admits an eight-dimensional manifold of periodic solutions called solitons

$$w(\pi) = w(\alpha, \Gamma, v_k, D_k) = e^{i(\Gamma + v\cdot x - t|v|^2 + \alpha^2 t)}\phi(x - 2tv - D, \alpha), \tag{0.2}$$

where $\phi(x, \alpha)$ is a positive ground state solution of the semilinear elliptic equation

$$-\Delta\phi + \alpha^2\phi = \phi^3. \tag{0.3}$$

We prove that in the neighborhood of the soliton manifold there exists a $\dot H^{1/2}$ real-analytic manifold $\mathcal{N}$ of asymptotically stable solutions of (0.1), meaning they are the sum of a moving soliton and a dispersive term.

Furthermore, we show that a solution starting on $\mathcal{N}$ remains on $\mathcal{N}$ for all positive time and for some finite negative time and that $\mathcal{N}$ is a centre-stable manifold for this equation. The proof of the nonlinear results is based on the method of modulation, introduced by Soffer and Weinstein and adapted by Schlag to the L^2-supercritical case. Novel elements include a different linearization and a Strichartz estimate for the time-dependent linear Schrödinger equation.

The main result depends on a spectral assumption concerning the absence of embedded eigenvalues in the continuous spectrum of the Hamiltonian.

1. Introduction

1.1. Main result.

Consider equation (0.1):

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi = 0.$$

For a parameter path $\pi = (v_k, D_k, \alpha, \Gamma)$ such that $\|\dot\pi\|_{L^1_t \cap L^\infty_t} < \infty$ and a positive solution $\phi(x, \alpha)$ of (0.3)

$$-\Delta\phi + \alpha^2\phi = \phi^3,$$

define the nonuniformly moving soliton $w_\pi(t)$ by

$$\begin{aligned} w_\pi(t)(x) &= e^{i\theta(x,t)}\phi(x - y(t), \alpha(t)) \\ \theta(x, t) &= v(t)\cdot x + \int_0^t (\alpha^2(s) - |v(s)|^2)\, ds + \Gamma(t) \\ y(t) &= 2\int_0^t v(s)\, ds + D(t). \end{aligned} \tag{1.1}$$

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Theorem 1.1 (Main result). *There exists a codimension-one real analytic manifold $\mathcal{N} \subset \dot H^{1/2}$, in a neighborhood of the soliton manifold, such that for any initial data $\psi(0) \in \mathcal{N}$, equation (0.1) has a global solution ψ. $\mathcal{N}$ and ψ have the following properties:*

1. *ψ is asymptotically stable, in the sense that it decomposes into a dispersive term r and a moving soliton $w_\pi(t)$, given by (1.1), which converges to a final state:*
$$\begin{aligned} \psi &= r + w_\pi(t) \\ \|\dot\pi\|_1 &\leq C\alpha(0)\|r(0)\|_{\dot H^{1/2}_x} \\ \|r\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} &\leq C\|r(0)\|_{\dot H^{1/2}_x} \leq C \min_{w \text{ is a soliton}} \|\psi(0) - w\|_{\dot H^{1/2}_x}. \end{aligned} \tag{1.2}$$
2. *ψ, r, and w_π depend real-analytically on the initial data $\psi(0)$.*
3. *The dispersive term r scatters like the solution of the free Schrödinger equation: there exists $r_0 \in \dot H^{1/2}$ such that*
$$r(t) = e^{it\Delta} r_0 + o_{\dot H^{1/2}}(1). \tag{1.3}$$
4. *ψ stays on $\mathcal{N}$ for infinite positive time and for finite negative time, meaning that $\mathcal{N}$ is invariant under the Hamiltonian flow.*
5. *$\mathcal{N}$ is the centre-stable manifold of the equation.*

This result depends on the absence of embedded eigenvalues within the continuous spectrum of the linearized Hamiltonian (Section 1.5). For a definition of centre-stable manifolds, the reader is referred to Section 1.3; for a definition of the norms involved in the statement of Theorem 1.1, one is referred to the Appendix.

A similar result holds going backward in time. The invariant manifold in that case is obtained by conjugating the one described in Theorem 1.1.

As a final matter concerning notation, we denote by C various constants that appear in the proof, not all equal.

1.2. **Background and history of the problem.** From a physical point of view, the nonlinear Schrödinger equation in $\mathbb{R}^3$ with cubic nonlinearity and the focusing sign (0.1) describes, to a first approximation, the self-focusing of optical beams due to the nonlinear increase of the refraction index. As such, the equation appeared for the first time in the physical literature in 1965, in [Kelley]. Equation (0.1) can also serve as a simplified model for the Schrödinger map equation and it arises as a limiting case of the Hartree equation, the Gross-Pitaevskii equation, or in other physical contexts.

More generally, consider the semilinear focusing Schrödinger equation in $\mathbb{R}^{n+1}$
$$i\partial_t \psi + \Delta\psi + |\psi|^p \psi = 0,\ \psi(0) \text{ given}. \tag{1.4}$$
It admits soliton solutions, $e^{it\alpha^2}\phi(x,\alpha)$, where
$$-\Delta\phi + \alpha^2\phi = \phi^{p+1}. \tag{1.5}$$
Important invariant quantities for this equation include the mass
$$M[\psi] = \int_{\mathbb{R}^n} |\psi(x,t)|^2\, dx \tag{1.6}$$
and the energy
$$E[\psi] = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla\psi(x,t)|^2 - \frac{2}{p+2}|\psi(x,t)|^{p+2}\, dx. \tag{1.7}$$

Equation (1.4) is invariant under the rescaling

$$\psi(x,t) \to \alpha^{2/p}\psi(\alpha x, \alpha^2 t). \tag{1.8}$$

We interpret this as meaning that (1.4) is $\dot{H}^{s_c}$-critical, for $s_c = n/2 - 2/p$. Of particular interest are the L^2 or mass-critical ($p = 4/n$) and the $\dot{H}^1$ or energy-critical ($p = 4/(n-2)$) cases.

Except for this introductory discussion of other authors' results, we always assume that $n = 3$ and $p = 2$, in which case (1.4) reduces to (0.1) and (1.5) to (0.3). In particular, (0.1) is $\dot{H}^{1/2}$-critical; in general, $p = 4/(n-1)$ is $\dot{H}^{1/2}$-critical.

Given a soliton solution (1.5) of the Schrödinger equation (1.4), a natural question concerns its stability under small perturbations. This issue has been addressed in the L^2-subcritical case by Cazenave and Lions [CaLi] and Weinstein [Wei2], [Wei3]. Their work addressed the question of orbital stability and introduced the method of modulation, which also figured in every subsequent result.

A first asymptotic stability result was obtained by Soffer–Weinstein [SoWe1], [SoWe2]. Further results belong to Pillet–Wayne [PiWa], Buslaev–Perelman [BuPe1], [BuPe2], [BuPe3], Cuccagna [Cuc1], [Cuc2], Rodnianski–Schlag–Soffer [RoScSo1], [RoScSo2], Tsai–Yau [TsYa1], [TsYa2], [TsYa3], Gang–Sigal [GaSi], and Cuccagna–Mizumachi [CuMi].

Grillakis, Shatah, and Strauss [GrShSt1], [GrShSt2] developed a general theory of soliton stability for Hamiltonian evolution equations, which, when applied to the Schrödinger equation, shows the dichotomy between the L^2-subcritical and critical or supercritical cases.

In the L^2-supercritical, H^1-subcritical case in $\mathbb{R}^3$, Schlag proved the existence of a codimension-one Lipschitz manifold of $W^{1,1} \cap H^1$ initial data that generate asymptotically stable solutions to (0.1). This was followed by more results in the same vein, such as Buslaev–Perelman [BuPe1], Krieger–Schlag [KrSc1], Cuccagna [Cuc2], Beceanu [Bec1], and Marzuola [Mar].

In the L^2-critical or supercritical case, negative energy $\langle x \rangle^{-1} L^2 \cap H^1$ initial data leads to solutions of (1.4) that blow up in finite time, due to the virial identity (see Glassey [Gla]). For a relaxation of this condition and a survey of results see [SuSu] and [Caz]. Berestycki–Cazenave [BeCa] showed that blow-up can occur for arbitrarily small perturbations of ground states (1.5). Recent results concerning this subject include Merle–Raphael [MeRa] and Krieger–Schlag [KrSc2].

Weinstein [Wei1] showed that all H^1 solutions ϕ of the mass-critical equation (1.4) of mass strictly less than that of the soliton, $M[\psi] < M[\phi]$, have global in time existence. Merle [Mer] classified threshold solutions ($M[\psi] = M[\phi]$), finding necessary and sufficient criteria for blow-up for H^1 data (namely, all blow-up solutions arise from transformations of the soliton), and global existence and scattering, for $\langle x \rangle^{-1} L^2 \cap H^1$ data.

A comparable result was obtained in 2006 by Kenig–Merle [KeMe] for the energy-critical equation (1.4) in the radial case. Namely, for radial $\dot{H}^1$ data ϕ of energy strictly less than that of the soliton, $E[\psi] < E[\phi]$, the following dichotomy takes place: if $\|\nabla \psi(0)\|_2 < \|\nabla \phi\|_2$, then ϕ exists globally and scatters, while if $\|\nabla \psi(0)\|_2 > \|\nabla \phi\|_2$ and $\psi(0) \in L^2$ then ψ blows up in finite time. In this regime, the equality $\|\nabla \psi(0)\|_2 = \|\nabla \phi\|_2$ cannot occur. The behavior of solutions at the energy threshold, $E[\psi] = E[\phi]$, was then classified by Duyckaerts–Merle [DuMe]: the same two cases are present, together with three more.

Following this approach, Holmer–Roudenko [HoRo], Duyckaerts–Holmer–Roudenko [DuHoRo], and Duyckaerts–Roudenko [DuRo] established corresponding results for the $\dot{H}^{1/2}$-critical equation (0.1). Their main findings may be summarized as follows:

Theorem 1.2. *Assume that ψ is a solution of (0.1) with*

$$M[\psi]E[\psi] - 2P[\psi]^2 \leq M[\phi]E[\phi], \tag{1.9}$$

where $\phi > 0$ is a soliton given by (0.3) and P is the momentum

$$P[\psi] = \int_{\mathbb{R}^3} \overline{\psi}(x,t) i \nabla \psi(x,t)\, dx. \tag{1.10}$$

Then one of the following holds:

(1) *If $M[\psi]\|\nabla\psi\|_2^2 - 2P[\psi]^2 < M[\phi]\|\nabla\phi\|_2^2$, then ψ exists globally and scatters or equals a special solution, ϕ_-, up to symmetries: Galilean coordinate changes, scaling, complex phase change, or conjugation.*
(2) *If $M[\psi]\|\nabla\psi\|_2^2 - 2P[\psi]^2 = M[\phi]\|\nabla\phi\|_2^2$, then ψ equals $e^{it}\phi(\cdot,1)$ up to symmetries.*
(3) *If $M[\psi]\|\nabla\psi\|_2^2 - 2P[\psi]^2 > M[\phi]\|\nabla\phi\|_2^2$ and $\psi \in \langle x\rangle^{-1}L^2$ is radial, then ψ blows up in finite time or must equal, up to symmetries, a special solution ϕ_+.*

The special solutions ϕ_- and ϕ_+ were defined by the aforementioned authors to have the following properties: $\phi_-(t)$ scatters as $t \to -\infty$ and converges at an exponential rate to a soliton solution as $t \to +\infty$, while $\phi_+(t)$ blows up in finite time for $t < 0$ and converges at an exponential rate to a soliton solution as $t \to +\infty$.

We explore the connection between their result, Theorem 1.2, and Theorem 1.1 in the subsequent remark.

Remark 1.3. *The boundary of the region*

$$B = \{\psi \mid M[\psi]E[\psi] - 2P[\psi]^2 \leq M[\phi]E[\phi]\}, \tag{1.11}$$

is not a smooth manifold. Theorem 1.1 shows that, in the neighborhood of the soliton manifold, B is contained between two transverse hypersurfaces. The boundary of the region B and each of the two hypersurfaces intersect along nine-dimensional manifolds.

The intersections are described by the special solutions $\phi_+(t)$ and $\phi_-(t)$, for sufficiently large t, subject to the symmetry transformations.

This remark fits with several other natural observations. Firstly, eliminating the soliton manifold (ϕ and its symmetry transformations — an eight-dimensional set) from B, an infinite-dimensional set, divides the latter into two disconnected components; this certainly could not happen if B were smooth.

Secondly, consider the functional that defines B,

$$F[\psi] = M[\psi]E[\psi] - 2P[\psi]^2. \tag{1.12}$$

Due to the extremizing property of the soliton ϕ in regard to the Gagliardo-Nirenberg-Sobolev inequality, the first differential of F at ϕ is identically zero:

$$\lim_{\epsilon\to 0} \frac{F[\phi + \epsilon h] - F[\phi]}{\epsilon} = 0. \tag{1.13}$$

Therefore, the tangent cone to B at ϕ is actually given by the sign of second differential of F, which is an indefinite quadratic form; hence the lack of smoothness.

The most directly relevant results to which Theorem 1.1 should be compared are those of Schlag [Sch], Beceanu [Bec1], and Cuccagna [Cuc2].

In [Sch], Schlag extended the method of modulation to the L^2-supercritical case and proved that in the neighborhood of each soliton there exists a codimension-one Lipschitz submanifold of $H^1(\mathbb{R}^3) \cap W^{1,1}(\mathbb{R}^3)$ such that initial data on the submanifold lead to global $H^1 \cap W^{1,\infty}$ solutions to (0.1), which decompose into a moving soliton and a dispersive term.

[Bec1] showed that for initial data in $\Sigma = \langle x \rangle^{-1} L^2 \cap H^1$, on a codimension one local Lipschitz manifold, there exists a global solution to (0.1) in the same space Σ. Furthermore, the manifold is identified as the centre-stable manifold for the equation in the space Σ (in particular, the solution stays on the manifold for some positive finite time).

Cuccagna [Cuc2] constructed asymptotically stable solutions for the one-dimensional mass-supercritical Schrödinger equation ((1.4) for $n = 1$, $5 < p < \infty$)

$$iu_t + u_{xx} + |u|^p u = 0,\ 5 < p < \infty, \tag{1.14}$$

starting from even H^1 initial data ($p = 5$ is the L^2-critical exponent in one dimension, while every exponent is H^1-subcritical). The set of solutions obtained was not endowed with a manifold structure.

Unlike previous results, the current one (Theorem 1.1) holds in a critical space, $\dot{H}^{1/2}$, for equation (0.1). Previously it was not known whether the asymptotically stable manifold exists in the critical norm or is a phenomenon related to using stronger norms than the critical one in the study of the equation. The current work puts this question to rest.

In addition, working in the critical space permits a series of improvements in the nature of results obtained. Firstly, we identify the asymptotically stable manifold as a centre-stable manifold for the equation in the sense of [BaJo]. Secondly, using the fact that the critical norm does not grow with time, we can prove that the manifold is globally in time invariant: solutions starting on the manifold exist globally and remain on the manifold, for all positive time.

Finally, showing that the centre-stable manifold is real analytic raises the issue of its analytic continuation beyond the immediate neighborhood of the soliton manifold. It becomes an interesting problem to identify the global object that corresponds to the local centre-stable manifold.

1.3. **The centre-stable manifold.** In 1983, Keller [Keller] considered the semilinear wave equation in $\mathbb{R}^N$ with damping

$$u_{tt} + au_t - \Delta u + f(u) = 0 \tag{1.15}$$

and a stationary solution given by

$$-\Delta u_0 + f(u_0) = 0. \tag{1.16}$$

Under some growth and smoothness conditions on f, namely

F1 $f(s) - ms = \tilde{f}(s) \in C^1(\mathbb{R})$, $m > 0$;
F2 $\tilde{f}(0) = \tilde{f}'(0) = 0$;
F3 $|\tilde{f}'(s)| \leq C(1 + |s|^{2/(N-2)})$;
F4 $F(s) = \int_0^s f(t)\,dt < 0$ for some $s > 0$,

Keller proved the existence of an infinite-dimensional invariant local Lipschitz manifold of H^1 solutions that approach u_0 as $t \to \infty$ and of a finite-dimensional invariant local Lipschitz manifold of solutions that approach u_0 as t goes to $-\infty$.

In 1989, Bates–Jones [BaJo] proved for a large class of semilinear equations that the space of solutions decomposes into an unstable and a centre-stable manifold. Their result is the following: consider a Banach space X and the semilinear equation

$$u_t = Au + f(u), \tag{1.17}$$

under the assumptions

H1 $A : X \to X$ is a closed, densely defined linear operator that generates a C_0 group.
H2 The spectrum of A decomposes into $\sigma(A) = \sigma_s(A) \cup \sigma_c(A) \cup \sigma_u(A)$ situated in the left half-plane, on the imaginary axis, and in the right half-plane respectively and $\sigma_s(A)$ and $\sigma_u(A)$ are bounded.
H3 The nonlinearity f is locally Lipschitz, $f(0) = 0$, and $\forall \epsilon > 0$ there exists a neighborhood of zero on which f has Lipschitz constant ϵ.

Moreover, let X^u, X^c, and X^s be the A-invariant subspaces corresponding to σ_u, σ_c, and respectively σ_s and let $S^c(t)$ be the evolution generated by A on X^c. Bates and Jones further assume that

C1-2 $dim X^u$, $dim X^s < \infty$.
C3 $\forall \rho > 0\ \exists M > 0$ such that $\|S^c(t)\| \leq M e^{\rho|t|}$.

Let Υ be the flow associated to the nonlinear equation. We call $\mathcal{N} \subset U$ t-invariant if, whenever $\Upsilon(s)v \in U$ for $s \in [0, t]$, $\Upsilon(s)v \in \mathcal{N}$ for $s \in [0, t]$.

Let W^u be the set of u for which $\Upsilon(t)u \in U$ for all $t < 0$ and decays exponentially as $t \to -\infty$. Also, consider the natural direct sum projection π^{cs} on $X^c \oplus X^s$.

Definition 1.1. *A* centre-stable *manifold $\mathcal{N} \subset U$ is a Lipschitz manifold with the property that $\mathcal{N}$ is t-invariant relative to U, $\pi^{cs}(\mathcal{N})$ contains a neighborhood of* 0 *in $X^c \oplus X^s$, and $\mathcal{N} \cap W^u = \{0\}$.*

The finding of [BaJo] is then

Theorem 1.4. *Under assumptions H1-H3 and C1-C3, there exists an open neighborhood U of zero such that W^u is a Lipschitz manifold which is tangent to X^u at* 0 *and there exists a centre-stable manifold $W^{cs} \subset U$ which is tangent to X^{cs}.*

Gesztesy, Jones, Latushkin, Stanislavova [GJLS] proved that Theorem 1.4 applies to the semilinear Schrödinger equation. More precisely, the authors established a spectral mapping theorem for the semigroup generated by linearizing the equation about the standing wave. In particular, this theorem shows that the spectral condition H2 implies the semigroup norm estimate C3. The local Lipschitz property of the nonlinearity is not addressed in [GJLS]; it holds on H^s for sufficiently large s.

While providing a very general answer to the problem, the global existence of solutions on the centre manifold is not the subject of [BaJo] and [GJLS]. The t-invariance property of [BaJo] means that a solution starting on the manifold remains there for as long as it stays small. However, one does not know the global in time behavior of the solutions. Once a solution on the centre-stable manifold leaves the specified neighborhood of zero, not even its existence is guaranteed any longer.

By contrast, the estimates used by Keller [KELLER] required a damping term, but his result held globally in time. Schlag [SCH] dispensed with the need for a damping term and proved a global asymptotic stability result, but the manifold that he constructed was not time-invariant.

The current work identifies a centre-stable manifold for (0.1) in the critical space for the equation $\dot{H}^{1/2}$ and shows that solutions starting on the manifold exist globally and remain on the manifold for all time.

1.4. **Setting and notations.** Equation (0.1) admits periodic solutions $e^{it\alpha^2}\phi(x,\alpha)$, where

$$\phi = \phi(x,\alpha) = \alpha\phi(\alpha x, 1) \tag{1.18}$$

is a solution of the semilinear elliptic equation (0.3)

$$-\Delta\phi + \alpha^2\phi = \phi^3.$$

In particular, we restrict our attention to positive L^2 solutions, called *ground states.* They are unique up to translation, radially symmetric, smooth, and exponentially decreasing. Their existence was proved by Berestycki–Lions in [BELI], who further showed that solutions are infinitely differentiable and exponentially decaying. Uniqueness was established by Coffman [COF] for the cubic and Kwong [KWO] and McLeod–Serrin [MCSE] for more general nonlinearities.

Equation (0.1) is invariant under Galilean coordinate transformations, rescaling, and changes of complex phase, which we collectively call symmetry transformations:

$$\mathfrak{g}(t)(f(x,t)) = e^{i(\Gamma+v\cdot x-t|v|^2)}\alpha f(\alpha(x-2tv-D),\alpha^2 t). \tag{1.19}$$

If $\psi(t)$ is a solution to the equation then so is $\mathfrak{g}(t)\psi(t)$, with initial data given by $\mathfrak{g}(0)\psi(0)$.

Applying these transformations to $e^{it}\phi(\cdot,1)$, the result is a wider eight-parameter family of solutions to (0.1)

$$\mathfrak{g}(t)(e^{it}\phi(x,1)) = e^{i(\Gamma+v\cdot x-t|v|^2+\alpha^2 t)}\phi(x-2tv-D,\alpha), \tag{1.20}$$

which we call solitons or standing waves.

In the sequel we denote by a capital letter the column vector consisting of a complex-valued function (written in lowercase) and its conjugate, e.g.

$$\Psi = \begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix},\ R = \begin{pmatrix} r \\ \overline{r}\end{pmatrix},\ \text{etc.} \tag{1.21}$$

On this space we consider the real-valued dot product

$$\langle F,G\rangle = \langle f,g\rangle + \langle \overline{f},\overline{g}\rangle = \int_{\mathbb{R}^3}\big(f(x)\overline{g}(x) + \overline{f}(x)g(x)\big)\,dx. \tag{1.22}$$

We leave it to the reader to check that our computations preserve this symmetry, in the sense that all column vectors that appear in the sequel comprise a function and its conjugate, meaning they are of the form $F = \begin{pmatrix} f \\ \overline{f}\end{pmatrix}$.

We look for solutions to (0.1) that get asymptotically close to the manifold of solitons. More precisely, we seek solutions of the form

$$\Psi = W_\pi(x,t) + R(x,t) = \begin{pmatrix} w_\pi(x,t) \\ \overline{w}_\pi(x,t)\end{pmatrix} + \begin{pmatrix} r(x,t) \\ \overline{r}(x,t)\end{pmatrix}, \tag{1.23}$$

where $W_\pi = \begin{pmatrix} w_\pi \\ \overline{w}_\pi \end{pmatrix}$ is a moving soliton and $R = \begin{pmatrix} r \\ \overline{r} \end{pmatrix}$ is a small correction term that disperses like the solution of the free Schrödinger equation as $t \to +\infty$.

We parametrize the moving soliton w_π by setting, as in (1.1),

$$\begin{aligned} w_\pi &= e^{i\theta(x,t)} \phi(x - y(t), \alpha(t)) \\ &= e^{i(\Gamma(t) + \int_0^t (\alpha^2(s) - |v(s)|^2)\, ds + v(t) x)} \phi(x - 2\textstyle\int_0^t v(s)\, ds - D(t), \alpha(t)). \end{aligned} \tag{1.24}$$

The quantities that appear in this formula,

$$\alpha(t),\ \Gamma(t),\ v(t) = (v_1(t), v_2(t), v_3(t)),\ \text{and } D(t) = (D_1(t), D_2(t), D_3(t)), \tag{1.25}$$

are called *modulation parameters* and $\pi = (\alpha, \Gamma, v, D)$ is called the *parameter* or *modulation path*. There are eight parameters in total, since v and D have three components each.

Due to the nonlinear interaction between w_π and r, the modulation parameters are not constant in general; they are time-dependent. However, in the course of the proof they are not allowed to vary too much. A minimal condition, which we impose henceforth, is that

$$\dot\alpha, \dot\Gamma, \dot v_k, \dot D_k \in L^1_t \tag{1.26}$$

and are small in this norm. We do not assume any stronger rate of decay. This implies that the modulation parameters converge as $t \to \infty$, at no particular rate, and that their range is contained within arbitrarily small intervals.

1.5. **Spectrum of the Hamiltonian and the spectral assumption.** The proof of Theorem 1.1 is based on a fixed point argument. Linearizing the equation around a moving soliton produces a time-dependent Hamiltonian of the following form, for a soliton w_{π^0} determined by a modulation path π^0 as in (1.24):

$$\mathcal{H}_{\pi^0}(t) = \begin{pmatrix} \Delta + 2|w_{\pi^0}(t)|^2 & (w_{\pi^0}(t))^2 \\ -(\overline{w}_{\pi^0}(t))^2 & -\Delta - 2|w_{\pi^0}(t)|^2 \end{pmatrix}. \tag{1.27}$$

We can always reduce the study of $i\partial_t + \mathcal{H}_{\pi^0}(t)$ to that of $i\partial_t + \mathcal{H}$, where

$$\mathcal{H} = \begin{pmatrix} \Delta - 1 + 2\phi^2(\cdot, 1) & \phi^2(\cdot, 1) \\ -\phi^2(\cdot, 1) & -\Delta + 1 - 2\phi^2(\cdot, 1) \end{pmatrix}. \tag{1.28}$$

The spectrum of this Hamiltonian has been extensively studied. In Section 2.2 we give a summary of the known facts. In addition, though, we must make the following *spectral assumption*:

Assumption A. *The Hamiltonian $\mathcal{H}$ has no embedded eigenvalues within its continuous spectrum.*

Such assumptions are routinely made in the proof of asymptotic stability results, as, for example, in [BuPe1], [Cuc1], [RoScSo2], or [Sch].

1.6. **Linear results.** In the process of establishing the main nonlinear result, Theorem 1.1, a crucial ingredient turns out to be a Strichartz-type estimate for the solution of the time-dependent linear Schrödinger equation

$$i\partial_t Z - iv(t)\nabla Z + A(t)\sigma_3 Z + \mathcal{H}Z = F,\ Z(0) \text{ given}. \tag{1.29}$$

Here

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix} + \begin{pmatrix} W_1 & W_2 \\ -W_2 & -W_1 \end{pmatrix},\ \mu > 0. \tag{1.30}$$

W_1 and W_2 are real-valued.

For the purpose of establishing the nonlinear result Theorem 1.1, we may as well consider only potentials V that are smooth and exponentially decaying.

We are mainly interested in Strichartz estimates

$$\|Z\|_{L^2_t L^6_x \cap L^\infty_t L^2_x} \leq C(\|Z(0)\|_2 + \|F\|_{L^2_t L^{6/5}_x + L^1_t L^2_x}). \tag{1.31}$$

Note that the Hamiltonian (1.28) is nonselfadjoint and time-dependent, leading to specific problems that are absent in the selfadjoint case.

In the scalar selfadjoint setting, Keel–Tao [KeTa] proved endpoint Strichartz estimates for the free Schrödinger and wave equations and introduced a general method for obtaining endpoint estimates (based on real interpolation) that has been useful in all similar contexts.

The issue of selfadjointness matters because one needs to reprove the usual dispersive estimates concerning the Schrödinger equation. They do not follow in the same manner as in the selfadjoint case, where, for example, the unitarity of the time evolution immediately implies the L^2 boundedness.

In [Sch], Schlag proved $L^1 \to L^\infty$ dispersive estimates for the Schrödinger equation with a nonselfadjoint Hamiltonian, as well as non-endpoint Strichartz estimates. Erdoĝan and Schlag [ErSc] proved L^2 bounds for the evolution as well. In [Bec1], endpoint Strichartz estimates in the nonselfadjoint case were obtained following the method of Keel and Tao. Finally, Cuccagna and Mizumatchi [CuMi] obtained the boundedness of the wave operators, from which all of the above can be inferred as a simple consequence.

In the linear setting, the most difficult to handle are terms of the form

$$(\alpha(t) - \alpha(\infty))\sigma_3 Z \text{ and } (v(t) - v(\infty))\nabla Z, \tag{1.32}$$

where Z is the solution. Instead of using Strichartz estimates to handle them, we make them part of the time-dependent Hamiltonian and thus avoid the issue altogether.

Our main result in this context, which improves upon and is distinct from previous estimates such as [RoSc], is the following:

Theorem 1.5 (see Theorem 3.9)**.** *Consider equation (1.29), for $\mathcal{H} = \mathcal{H}_0 + V$ as in (1.30) and a potential V not necessarily real-valued:*

$$i\partial_t Z - iv(t)\nabla Z + A(t)\sigma_3 Z + \mathcal{H}Z = F,\ Z(0) \text{ given},$$

$$\mathcal{H} = \begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix} + \begin{pmatrix} W_1 & W_2 \\ -W_2 & -W_1 \end{pmatrix}.$$

Assume that $V = V_1 V_2$ and V_1, V_2 are sufficiently smooth and decaying:

$$\langle x \rangle^n \partial^m V_j \in L^\infty \tag{1.33}$$

for every m, $n \leq N$ and some sufficiently large N. Also assume that $\|A\|_\infty$ and $\|v\|_\infty$ are sufficiently small (in a manner that depends on V) and $\sigma(\mathcal{H}_0)$ contains no exceptional values of $\mathcal{H}$. Then

$$\|P_c Z\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} \leq C\Big(\|R(0)\|_{\dot H^{1/2}} + \|F\|_{L^1_t \dot H^{1/2}_x + L^2_t \dot W^{1/2,6/5}_x}\Big). \tag{1.34}$$

The decay assumptions used here are not sharp. P_c is the projection on the continuous spectrum of $\mathcal{H}$. More detailed statements of these results and remarks concerning them follow within the paper. For statements that hold under sharp or almost sharp conditions, see [Bec2] and [Bec3].

2. The Nonlinear Result

2.1. **Deriving the linearized equation.** Substituting $\psi = w_\pi + r$ in the original equation (0.1),

$$i\partial_t(w_\pi + r) + \Delta(w_\pi + r) + (\overline{w}_\pi + \overline{r})(w_\pi + r)^2 = 0. \tag{2.1}$$

In keeping with (1.24), the soliton w_π is described by

$$w_\pi(t) = e^{i(\Gamma(t)+\int_0^t (\alpha^2(s)-|v(s)|^2)\, ds+v(t)\cdot x)}\phi\big(x - 2\int_0^t v(s)\, ds - D(t), \alpha(t)\big).$$

Note that $w_\pi(t)$ depends on the values π takes on $[0, t]$. Indeed, if we define

$$w(\pi) = w(\alpha, \Gamma, v, D) = e^{i(\Gamma+v\cdot x)}\phi\big(x - D, \alpha\big), \tag{2.2}$$

then $w_\pi(t) \neq w(\pi(t))$; in fact,

$$w_\pi(t) = w\left(\alpha(t), \Gamma(t) + \int_0^t (\alpha^2(s) - |v(s)|^2)\, ds, v(t), D(t) + 2\int_0^t v(s)\, ds\right). \tag{2.3}$$

Expanding the equation accordingly, note that

$$\partial_t w_\pi = (\dot\Gamma + \alpha^2 - v^2)\partial_\Gamma w_\pi + \dot\alpha\partial_\alpha w_\pi + \dot v\partial_v w_\pi - (2v + \dot D)\partial_D w_\pi \tag{2.4}$$

and

$$\begin{aligned}\Delta w_\pi &= \Delta e^{i\theta(x,t)}\phi(x - y(t), \alpha(t)) + 2\nabla e^{i\theta(x,t)}\nabla\phi(x - y(t), \alpha(t)) +\\ &\quad + e^{i\theta(x,t)}\Delta\phi(x - y(t), \alpha(t))\\ &= (\alpha^2 - v^2)w_\pi + 2iv\nabla w_\pi - |w_\pi|^2 w_\pi.\end{aligned} \tag{2.5}$$

Here we used the following notation: given a soliton $w = w(\pi)$ as in (2.2), $\partial_\Gamma w$, $\partial_\alpha w$, $\partial_{D_k} w$, and $\partial_{v_k} w$ are the partial derivatives of w with respect to these parameters and span the tangent space to the soliton manifold at the point w:

$$\partial_\Gamma w = iw,\ \partial_\alpha w = \partial_\alpha w,\ \partial_{D_k} w = \partial_{x_k} w,\ \partial_{v_k} w = ix_k w. \tag{2.6}$$

We use these partial derivatives to define the differential $d_\pi w$ of the map $\pi \mapsto w = w(\pi)$. The differential $d_\pi w$ maps a vector $\delta\pi = (\delta\alpha, \delta\Gamma, \delta v_k, \delta D_k)$ in the tangent space at π to a vector in the tangent space at w:

$$(d_\pi w)\delta\pi = (\delta\Gamma)\partial_\Gamma w + (\delta\alpha)\partial_\alpha w + (\delta D_k)\partial_{D_k} w + (\delta v_k)\partial_{v_k} w. \tag{2.7}$$

This expansion results in the cancellation of the main term involving w_π. The equation concerning r becomes

$$i\partial_t r + \Delta r + i(\partial_\pi w_\pi)\dot\pi + (|r|^2 r + r^2\overline{w}_\pi + 2|r|^2 w_\pi + 2r|w_\pi|^2 + \overline{r}w_\pi^2) = 0. \tag{2.8}$$

Here $d_\pi w_\pi$ is the differential evaluated at the point w_π. Since as noted previously $w_\pi \neq w(\pi)$, $(d_\pi w_\pi)\dot\pi$ is not the same as the time derivative of w_π.

We regard equation (2.8) as a Schrödinger equation for r. By conjugating, we obtain an equivalent equation for $\overline{r}$:

$$i\partial_t \overline{r} - \Delta \overline{r} + i\overline{d_\pi w_\pi}\dot{\pi} - (|r|^2\overline{r} + (\overline{r})^2 w_\pi + 2|r|^2\overline{w}_\pi + 2\overline{r}|w_\pi|^2 + r(\overline{w}_\pi)^2) = 0. \quad (2.9)$$

Since both equations involve both r and $\overline{r}$, it is most convenient to solve them together as a system or, rather, to see the pair of equations as just one equation concerning the column vector $R = \begin{pmatrix} r \\ \overline{r} \end{pmatrix}$.

Adopting this point of view, the main terms assemble into a time-dependent, nonselfadjoint two-by-two *matrix* Hamiltonian

$$\mathcal{H}_\pi(t) = \begin{pmatrix} \Delta + 2|w_\pi(t)|^2 & w_\pi(t)^2 \\ -\overline{w}_\pi(t)^2 & -\Delta - 2|w_\pi(t)|^2 \end{pmatrix} = \Delta\sigma_3 + V_\pi(t), \quad (2.10)$$

whereas the other terms are better treated as the homogenous right-hand side of the equation:

$$F = \begin{pmatrix} i(d_\pi w_\pi)\dot{\pi} + |z|^2 z + z^2\overline{w}_\pi + 2|z|^2 w_\pi \\ i\,\overline{d_\pi w_\pi}\dot{\pi} - |z|^2\overline{z} - (\overline{z})^2 w_\pi - 2|z|^2\overline{w}_\pi \end{pmatrix}. \quad (2.11)$$

Given a $\mathbb{C}^2$-valued soliton $W = \begin{pmatrix} w \\ \overline{w} \end{pmatrix}$, we also introduce the partial derivatives $\partial_f W = \begin{pmatrix} \partial_f w \\ \overline{\partial_f w} \end{pmatrix}$, for $f \in \{\alpha, \Gamma, v_k, D_k\}$:

$$\partial_\Gamma W = \begin{pmatrix} iw \\ -i\overline{w} \end{pmatrix} = i\sigma_3 W,\ \partial_\alpha W = \partial_\alpha W,\ \partial_{D_k} W = \partial_{x_k} W,\ \partial_{v_k} W = i\sigma_3 x_k W. \quad (2.12)$$

As a reminder, σ_3 is one of the Pauli matrices:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.13)$$

We use these to define the differential $d_\pi W(\pi)$ of the map

$$\begin{aligned} \pi \mapsto W = W(\pi) &= \begin{pmatrix} w(\pi) \\ \overline{w}(\pi) \end{pmatrix}, \\ (d_\pi W(\pi))\delta\pi &= (\delta\Gamma)\partial_\Gamma W + (\delta\alpha)\partial_\alpha W + (\delta D_k)\partial_{D_k} W + (\delta v_k)\partial_{v_k} W. \end{aligned} \quad (2.14)$$

Thus, the first half of F is simply given by the differential

$$\begin{pmatrix} i(d_\pi w_\pi)\dot{\pi} \\ i\,\overline{d_\pi w_\pi}\dot{\pi} \end{pmatrix} = i(d_\pi W_\pi)\dot{\pi}. \quad (2.15)$$

We denote the remaining nonlinear term by

$$N(R, W_\pi) = \begin{pmatrix} -|r|^2 r - r^2\overline{w}_\pi - 2|r|^2 w_\pi \\ |r|^2\overline{r} + \overline{r}^2 w_\pi + 2|r|^2\overline{w}_\pi \end{pmatrix}. \quad (2.16)$$

In vector form, the equation fulfilled by R becomes

$$i\partial_t R - \mathcal{H}_\pi(t)R = F(t),\ F = -i(d_\pi W_\pi)\dot{\pi} + N(R, W_\pi). \quad (2.17)$$

To this equation concerning R we join the *modulation equations* that determine the path π. For future convenience, define the cotangent vectors

$$\begin{aligned} \Xi_\alpha(W) &= i\sigma_3\partial_\Gamma W, & \Xi_\Gamma(W) &= i\sigma_3\partial_\alpha W, \\ \Xi_{v_k}(W) &= i\sigma_3\partial_{D_k} W, & \Xi_{D_k}(W) &= i\sigma_3\partial_{v_k} W. \end{aligned} \quad (2.18)$$

Also consider the (second) differential $d_\pi \Xi_f(W)$, naturally defined as the differential of the map $\pi \mapsto \Xi_f(W(\pi))$, for $f \in \{\alpha, \Gamma, v_k, D_k\}$.

At each time t and for all $f \in \{\alpha, \Gamma, v_k, D_k\}$ we impose the *orthogonality condition*

$$\langle R(t), \Xi_f(W_\pi(t))\rangle = 0. \tag{2.19}$$

This condition arises as follows: given parameters $\pi = (\alpha, \Gamma, v_k, D_k)$, let

$$\begin{aligned} w = w(\pi) &= e^{i(\Gamma + v\cdot x)}\phi(x - D, \alpha), \\ W = W(\pi) &= \begin{pmatrix} w \\ \overline{w} \end{pmatrix}, \\ \mathcal{H}(W) &= \begin{pmatrix} \Delta + 2|w|^2 & w^2 \\ -\overline{w}^2 & -\Delta - 2|w|^2 \end{pmatrix} + 2iv\nabla - (\alpha^2 - |v|^2)\sigma_3. \end{aligned} \tag{2.20}$$

The Hamiltonian $\mathcal{H}(W)$, naturally associated to each soliton W (see Section 2.2), will play an important role in the sequel.

Note that $\mathcal{H}(W_\pi(t)) = \mathcal{H}_\pi(t) + 2iv(t)\nabla - (\alpha(t)^2 - |v(t)|^2)\sigma_3$.

Condition (2.19) is then equivalent to asking that R should not live in the zero eigenspace of the time-dependent Hamiltonian $\mathcal{H}(W_\pi(t))$.

Taking the derivative in (2.19), it translates into the following modulation equations.

Lemma 2.1 (The modulation equations)**.**

$$\begin{aligned} \dot\alpha &= 4\alpha\|W_\pi\|_2^{-2}(\langle R, (d_\pi \Xi_\alpha(W_\pi))\dot\pi\rangle - i\langle N(R, W_\pi), \Xi_\alpha(W_\pi)\rangle) \\ \dot\Gamma &= 4\alpha\|W_\pi\|_2^{-2}(\langle R, (d_\pi \Xi_\Gamma(W_\pi))\dot\pi\rangle - i\langle N(R, W_\pi), \Xi_\Gamma(W_\pi)\rangle) \\ \dot v_k &= 2\|W_\pi\|_2^{-2}(\langle R, (d_\pi \Xi_{v_k}(W_\pi))\dot\pi\rangle - i\langle N(R, W_\pi), \Xi_{v_k}(W_\pi)\rangle) \\ \dot D_k &= 2\|W_\pi\|_2^{-2}(\langle R, (d_\pi \Xi_{D_k}(W_\pi))\dot\pi\rangle - i\langle N(R, W_\pi), \Xi_{D_k}(W_\pi)\rangle). \end{aligned} \tag{2.21}$$

Proof. Begin by observing that for every soliton W

$$\begin{aligned} \langle \partial_\alpha W, \Xi_f(W)\rangle &= \frac{1}{4\alpha}\|W\|_2^2 \text{ if } f = \alpha \text{ and zero otherwise} \\ \langle \partial_\Gamma W, \Xi_f(W)\rangle &= \frac{1}{4\alpha}\|W\|_2^2 \text{ if } f = \Gamma \text{ and zero otherwise} \\ \langle \partial_{D_k} W, \Xi_f(W)\rangle &= \frac{1}{2}\|W\|_2^2 \text{ if } f = D_k \text{ and zero otherwise} \\ \langle \partial_{v_k} W, \Xi_f(W)\rangle &= \frac{1}{2}\|W\|_2^2 \text{ if } f = v_k \text{ and zero otherwise.} \end{aligned} \tag{2.22}$$

Furthermore,

$$\begin{aligned} \mathcal{H}^*(W)\Xi_\alpha(W) = 0, &\quad \mathcal{H}^*(W)\Xi_\Gamma(W) = -2i\Xi_\alpha, \\ \mathcal{H}^*(W)\Xi_{v_k}(W) = 0, &\quad \mathcal{H}^*(W)\Xi_{D_k}(W) = -2i\Xi_{v_k}. \end{aligned} \tag{2.23}$$

Finally,

$$\partial_t \Xi_f(W_\pi) = (d_\pi \Xi_f(W_\pi))\dot\pi + (\mathcal{H}^*(W_\pi) - \mathcal{H}^*_\pi)\Xi_f(W_\pi). \tag{2.24}$$

Then, in the equality

$$\langle R, \partial_t \Xi_f(W_\pi)\rangle = -\langle \partial_t R, \Xi_f\rangle, \tag{2.25}$$

we replace $\partial_t R$ by its expression (2.17) and arrive at (2.21). $\square$

Let

$$
\begin{aligned}
L_\pi R = & 4\alpha \sum_{f\in\alpha,\Gamma} \|W_\pi\|_2^{-2}\langle R, (d_\pi \Xi_f(W_\pi))\dot\pi\rangle \partial_f W_\pi \\
& +2 \sum_{f\in\{v_k,D_k\}} \|W_\pi\|_2^{-2}\langle R, (d_\pi \Xi_f(W_\pi))\dot\pi\rangle \partial_f W_\pi
\end{aligned} \tag{2.26}
$$

and

$$
\begin{aligned}
N_\pi(R, W_\pi) = & 4\alpha \sum_{f\in\{\alpha,\Gamma\}} \|W_\pi\|_2^{-2} i\langle N(R, W_\pi), \Xi_f(W_\pi)\rangle \partial_f W_\pi \\
& +2 \sum_{f\in\{v_k,D_k\}} \|W_\pi\|_2^{-2} i\langle N(R, W_\pi), \Xi_f(W_\pi)\rangle \partial_f W_\pi.
\end{aligned} \tag{2.27}
$$

The modulation equations can then be rewritten as

$$
(d_\pi W_\pi)\dot\pi = L_\pi R - iN_\pi(R, W_\pi). \tag{2.28}
$$

$L_\pi R$ represents the part that is linear in R and $N_\pi(R, W)$ represents the nonlinear component $\langle N(R, W), \Xi_f(W_\pi)\rangle$.

As an aside, using the notation of Section 2.2, let $P_0(W_\pi)$ be the zero spectrum projection of the Hamiltonian corresponding to W_π. We recognize

$$
N_\pi(R, W_\pi) = P_0(W_\pi(t))N(R, W_\pi) \tag{2.29}
$$

and a similar, but more complicated, expression for $L_\pi R$.

At this point we linearize the equation. We use an auxiliary function for all quadratic and cubic terms and do the same in regard to the soliton. We only keep linear, first-order terms in the unknowns R and π for which we solve the equation, so that the equation becomes linear in R and π and quadratic and cubic in the terms involving R^0 and π^0.

We introduce an auxiliary function R^0 and an auxiliary modulation path π^0. Starting with π^0, we construct the moving soliton w_{π^0}, its partial derivatives $\partial_f w_{\pi^0}$, cotangent vectors $\Xi_f(W_{\pi^0})$, the differentials $d_\pi W_{\pi^0}$ and $d_\pi \Xi_f(W_{\pi^0})$, etc., all following previous definitions:

$$
\begin{aligned}
\pi^0 &= (\alpha^0, \Gamma^0, v^0, D^0), \\
w_{\pi^0} &= e^{i(\Gamma^0(t)+\int_0^t ((\alpha^0)^2(s)-|v^0|^2(s))\,ds+v^0(t)\cdot x)}\phi(x - 2\textstyle\int_0^t v^0(s)\,ds - D^0(t), \alpha^0(t)), \\
W_{\pi^0} &= \begin{pmatrix} w_{\pi^0} \\ \overline{w}_{\pi^0} \end{pmatrix},
\end{aligned} \tag{2.30}
$$

as well as

$$
\begin{aligned}
\partial_\Gamma w_{\pi^0} = iw_{\pi^0},\ \partial_\alpha w_{\pi^0} &= \partial_\alpha w_{\pi^0},\ \partial_{D_k} w_{\pi^0} = \partial_{x_k} w_{\pi^0},\ \partial_{v_k} w_{\pi^0} = ix_k w_{\pi^0}, \\
\partial_f W_{\pi^0} &= \begin{pmatrix} \partial_f w_{\pi^0} \\ \partial_f \overline{w}_{\pi^0} \end{pmatrix}, \\
\Xi_\alpha(W_{\pi^0}) &= i\sigma_3 \partial_\Gamma W_{\pi^0}, \Xi_\Gamma(W_{\pi^0}) = i\sigma_3\partial_\alpha W_{\pi^0}, \\
\Xi_{v_k}(W_{\pi^0}) &= i\sigma_3\partial_{D_k} W_{\pi^0}, \Xi_{D_k}(W_{\pi^0}) = i\sigma_3\partial_{v_k} W_{\pi^0}.
\end{aligned} \tag{2.31}
$$

Lemma 2.2. *ψ is a solution of (0.1) if and only if*

$$
\Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} = W_{\pi^0} + R, \tag{2.32}
$$

W_{π^0} is given by (2.30), and (R, π) is a fixed point of the map $(R^0, \pi^0) \mapsto (R, \pi)$, where R and π solve the following system of linear equations:

$$\begin{aligned}
&i\partial_t R + \mathcal{H}_{\pi^0}(t) R = F,\ F = -iL_{\pi^0} R + N(R^0, W_{\pi^0}) - N_{\pi^0}(R^0, W_{\pi^0}) \\
&\dot f = 4\alpha^0 \|W_{\pi^0}\|_2^{-2} (\langle R, (d_\pi \Xi_f(W_{\pi^0}))\dot\pi^0\rangle - i\langle N(R^0, W_{\pi^0}), \Xi_f(W_{\pi^0})\rangle), f \in \{\alpha, \Gamma\} \\
&\dot f = 2\|W_{\pi^0}\|_2^{-2} (\langle R, (d_\pi \Xi_f(W_{\pi^0}))\dot\pi^0\rangle - i\langle N(R^0, W_{\pi^0}), \Xi_f(W_{\pi^0})\rangle), f \in \{v_k, D_k\}.
\end{aligned} \tag{2.33}$$

Here

$$\begin{aligned}
\mathcal{H}_{\pi^0}(t) &= \begin{pmatrix} \Delta + 2|w_{\pi^0}(t)|^2 & (w_{\pi^0}(t))^2 \\ -(\overline{w}_{\pi^0}(t))^2 & -\Delta - 2|w_{\pi^0}(t)|^2 \end{pmatrix}, \\
N(R^0, W_{\pi^0}) &= \begin{pmatrix} -|r^0|^2 r^0 - (r^0)^2 \overline{w}_{\pi^0} - 2|r^0|^2 w_{\pi^0} \\ |r^0|^2 \overline{r^0} + (\overline{r^0})^2 w_{\pi^0} + 2|r^0|^2 \overline{w}_{\pi^0} \end{pmatrix}, \\
L_{\pi^0} R &= 4\alpha^0 \sum_{f \in \alpha, \Gamma} \|W_{\pi^0}\|_2^{-2} \langle R, (d_\pi \Xi_f(W_{\pi^0}))\dot\pi^0\rangle \partial_f W_{\pi^0} \\
&\quad + 2 \sum_{f \in \{v_k, D_k\}} \|W_{\pi^0}\|_2^{-2} \langle R, (d_\pi \Xi_f(W_{\pi^0}))\dot\pi^0\rangle \partial_f W_{\pi^0}, \\
N_{\pi^0}(R^0, W_{\pi^0}) &= 4\alpha^0 \sum_{f \in \{\alpha, \Gamma\}} \|W_{\pi^0}\|_2^{-2} i\langle N(R^0, W_{\pi^0}), \Xi_f(W_{\pi^0})\rangle \partial_f W_{\pi^0} \\
&\quad + 2 \sum_{f \in \{v_k, D_k\}} \|W_{\pi^0}\|_2^{-2} i\langle N(R^0, W_{\pi^0}), \Xi_f(W_{\pi^0})\rangle \partial_f W_{\pi^0}.
\end{aligned} \tag{2.34}$$

Solving the system (2.33) in a suitable space, with a view toward applying a fixed point theorem to conclude that $(R^0, \pi^0) = (R, \pi)$, is our main objective.

Proof. We substitute R^0 for R and π^0 for π into all higher order terms of (2.17) and (2.21). Following this substitution, the nonlinear Schrödinger equation (2.17) transforms into the linear equation, in both R and $\dot\pi$,

$$i\partial_t R - \mathcal{H}_{\pi^0}(t) R = F, \tag{2.35}$$

where

$$\begin{aligned}
F &= -i(d_\pi W_{\pi^0})\dot\pi + N(R^0, W_{\pi^0}), \\
(d_\pi W_{\pi^0})\dot\pi &= \dot\Gamma \partial_\Gamma W_{\pi^0} + \dot\alpha \partial_\alpha W_{\pi^0} + \dot D_k \partial_{D_k} W_{\pi^0} + \dot v_k \partial_{v_k} W_{\pi^0}.
\end{aligned} \tag{2.36}$$

The orthogonality condition we impose in the linear setting is

$$\langle R(t), \Xi_f(W_{\pi^0}(t))\rangle = 0,\ f \in \{\alpha, \Gamma, v_k, D_k\}. \tag{2.37}$$

By taking the derivative, in a manner entirely similar to that of Lemma 2.1 we obtain the linearized modulation equations: (2.21) become

$$\begin{aligned}
\dot\alpha &= 4\alpha^0 \|W_{\pi^0}\|_2^{-2} (\langle R, (d_\pi \Xi_\alpha(W_{\pi^0}))\dot\pi^0\rangle - i\langle N(R^0, \pi^0), \Xi^0_\alpha\rangle) \\
\dot\Gamma &= 4\alpha^0 \|W^0\|_2^{-2} (\langle R, (d_\pi \Xi_\Gamma(W_{\pi^0}))\dot\pi^0\rangle - i\langle N(R^0, \pi^0), \Xi^0_\Gamma\rangle) \\
\dot v_k &= 2\|W^0\|_2^{-2} (\langle R, (d_\pi \Xi_{v_k}(W_{\pi^0}))\dot\pi^0\rangle - i\langle N(R^0, \pi^0), \Xi^0_{v_k}\rangle) \\
\dot D_k &= 2\|W^0\|_2^{-2} (\langle R, (d_\pi \Xi_{D_k}(W_{\pi^0}))\dot\pi^0\rangle - i\langle N(R^0, \pi^0), \Xi^0_{D_k}\rangle).
\end{aligned} \tag{2.38}$$

The identity (2.28) translates into

$$(d_\pi W_{\pi^0})\dot\pi = L_{\pi^0} R - iN_{\pi^0}(R^0, \pi^0). \tag{2.39}$$

Finally, we collect together (2.35) and (2.38) and replace $(d_\pi W_{\pi^0})\dot\pi$ on the right-hand side of (2.35) by its expression (2.28) in order to arrive at (2.33). $\square$

2.2. **Spectral considerations.** Given parameters $\pi = (\alpha, \Gamma, v_k, D_k)$, define the soliton

$$\begin{aligned} w &= w(\pi) = e^{i(\Gamma + v\cdot x)}\phi(x - D, \alpha), \\ W &= W(\pi) = \begin{pmatrix} w \\ \overline{w} \end{pmatrix} \end{aligned} \tag{2.40}$$

and consider the associated Hamiltonian

$$\mathcal{H}(W) = \mathcal{H}(\alpha, \Gamma, v, D) = \begin{pmatrix} \Delta + 2|w|^2 & w^2 \\ -\overline{w}^2 & -\Delta - 2|w|^2 \end{pmatrix} + 2iv\nabla - (\alpha^2 - |v|^2)\sigma_3. \tag{2.41}$$

By rescaling and conjugating by $e^{i(xv+\Gamma)\sigma_3}$ as well as by a translation, one sees that all these operators are in fact conjugate, up to a constant factor of α^2:

$$\mathrm{Dil}_{1/\alpha} e^{-D\nabla} T_D e^{-i(xv+\Gamma)\sigma_3} \mathcal{H}(\alpha, \Gamma, v, D) e^{i(xv+\Gamma)\sigma_3} e^{D\nabla} \mathrm{Dil}_\alpha = \alpha^2 \mathcal{H}(1, 0, 0, 0). \tag{2.42}$$

Therefore all have the same spectrum up to dilation and have similar spectral properties; thus, it suffices to study $\mathcal{H} = \mathcal{H}(1, 0, 0, 0)$:

$$\mathcal{H} = \begin{pmatrix} \Delta - 1 + 2\phi^2 & \phi^2 \\ -\phi^2 & -\Delta + 1 - 2\phi^2 \end{pmatrix}. \tag{2.43}$$

We restate the known facts about the spectrum of $\mathcal{H}$. As proved by Buslaev, Perelman [BuPe1] and also Rodnianski, Schlag, Soffer in [RoScSo2], under fairly general assumptions, $\sigma(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$ and is symmetric with respect to the coordinate axes and all eigenvalues are simple with the possible exception of 0. Furthermore, by Weyl's criterion $\sigma_{ess}(\mathcal{H}) = (-\infty, -1] \cup [1, +\infty)$.

Grillakis, Shatah, Strauss [GrShSt1] and Schlag [Sch] showed that there is only one pair of conjugate imaginary eigenvalues $\pm i\sigma$ and that the corresponding eigenvectors decay exponentially. For the decay see Hundertmark, Lee [HuLe]. The pair of conjugate imaginary eigenvalues $\pm i\sigma$ reflects the L^2-supercritical nature of the problem.

The generalized eigenspace at 0 arises due to the symmetries of the equation, which is invariant under Galilean coordinate changes, phase changes, and scaling. It is relatively easy to see that each of these symmetries gives rise to a generalized eigenvalue of the Hamiltonian $\mathcal{H}$ at 0, but proving the converse is much harder and was done by Weinstein in [Wei2], [Wei3].

Schlag [Sch] showed, using ideas of Perelman [Per], that if the operators

$$L_\pm = -\Delta + 1 - 2\phi^2(\cdot, 1) \mp \phi^2(\cdot, 1) \tag{2.44}$$

that arise by conjugating $\mathcal{H}$ with $\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ have no eigenvalue in $(0, 1]$ and no resonance at 1, then the real discrete spectrum of $\mathcal{H}$ is $\{0\}$ and the edges ± 1 are neither eigenvalues nor resonances. A work of Demanet, Schlag [DeSc] verified numerically that the scalar operators meet these conditions. Therefore, there are no eigenvalues in $[-1, 1]$ and ± 1 are neither eigenvalues nor resonances for $\mathcal{H}$.

Furthermore, the method of Agmon [Agm], adapted to the matrix case, enabled Erdogan–Schlag [ErSc] and independently [CuPeVo] to prove that any resonances embedded in the interior of the essential spectrum (that is, in $(-\infty, -1) \cup (1, \infty)$) have to be eigenvalues, under very general assumptions.

Under the spectral Assumption A we now have a complete description of the spectrum of $\mathcal{H}$. It consists of a pair of conjugate purely imaginary eigenvalues $\pm i\sigma$, a generalized eigenspace at 0, and the essential spectrum $(-\infty, -1] \cup [1, \infty)$.

Following [Sch], let $F^\pm$ be the eigenfunctions of $\mathcal{H}$ corresponding to $\pm i\sigma$; then there exists $f^+ \in L^2$ such that

$$F^+ = \begin{pmatrix} f^+ \\ \overline{f}^+ \end{pmatrix},\ F^- = \overline{F}^+ = \begin{pmatrix} \overline{f}^+ \\ f^+ \end{pmatrix}. \tag{2.45}$$

Due to the symmetry

$$\sigma_3 \mathcal{H} \sigma_3 = \mathcal{H}^* \tag{2.46}$$

of the operator, the respective eigenfunctions of $\mathcal{H}^*$ are $\sigma_3 F^\pm$. Then, the imaginary spectrum projection is given by

$$P_{im} = P_+ + P_-,\ P_\pm = \langle \cdot, i\sigma_3 F^\mp \rangle F^\pm \tag{2.47}$$

up to a constant; the constant becomes 1 after the normalization

$$\int_{\mathbb{R}^3} \operatorname{Re} f^+(x) \operatorname{Im} f^+(x)\, dx = -1/2. \tag{2.48}$$

It helps in the proof to exhibit the discrete eigenspaces of $\mathcal{H}(W)$. Denote by $F^\pm(W)$ the eigenfunctions of $\mathcal{H}(W)$ corresponding to the $\pm i\sigma(W)$ eigenvalues. Even though there is no explicit form of the imaginary eigenvectors, Schlag [Sch] proved in a more general setting that $F^\pm(W)$, L^2 normalized, and σ are locally Lipschitz continuous as a function of W and that $F^\pm(W)$ are exponentially decaying.

More is true in the case under consideration. Since the operators $\mathcal{H}(W)$ are conjugate up to a constant, the dependence of $F^\pm(W)$ and σ on the parameters can be made explicit:

$$\begin{aligned} F^\pm(W) &= e^{i(xv+\Gamma)\sigma_3} e^{D\nabla} \mathrm{Dil}_\alpha F^\pm, \\ \sigma(W) &= \alpha^2 \sigma. \end{aligned} \tag{2.49}$$

Also observe that $\partial_f W$, where $f \in \{\alpha, \Gamma, v_k, D_k\}$, are the generalized eigenfunctions of $\mathcal{H}(W)$ at zero and $\Xi_f(W)$, defined as in (2.18), fulfill the same role for $\mathcal{H}(W)^*$.

We then express the Riesz projections corresponding to the three components of the spectrum of $\mathcal{H}(W)$ as

$$P_{im}(W) = P_+(W) + P_-(W),\ P_\pm(W) = \alpha^{-3} \langle \cdot, i\sigma_3 F^\mp(W) \rangle F^\pm(W), \tag{2.50}$$

$$\begin{aligned} P_0(W) = {} & 4\alpha \|W\|_2^{-2} (\langle \cdot, \Xi_\alpha \rangle \partial_\alpha W + \langle \cdot, \Xi_\Gamma \rangle \partial_\Gamma W) + \\ & + 2\|W\|_2^{-2} \sum_k (\langle \cdot, \Xi_{v_k} \rangle \partial_{v_k} W + \langle \cdot, \Xi_{D_k} \rangle \partial_{D_k} W), \end{aligned} \tag{2.51}$$

and

$$P_c(W) = 1 - P_{im}(W) - P_0(W). \tag{2.52}$$

2.3. **The fixed point argument: stability.** Consider a small neighborhood of a fixed soliton $w_0 = w(\pi_0)$, determined by parameters $\pi_0 = (\alpha_0, \Gamma_0, v_{k0}, D_{k0})$:

$$\begin{aligned} W_0 &= \begin{pmatrix} w_0 \\ \overline{w}_0 \end{pmatrix}, \\ w_0 &= e^{i(\Gamma_0 + v_0 \cdot x)} \phi(x - D_0, \alpha_0). \end{aligned} \tag{2.53}$$

As in Section 2.2, W_0 has an associated Hamiltonian $\mathcal{H}(W_0)$ of the form (2.41)

$$\mathcal{H}(W_0) = \begin{pmatrix} \Delta + 2|w_0|^2 & w_0^2 \\ -\overline{w}_0^2 & -\Delta - 2|w_0|^2 \end{pmatrix} + 2iv_0\nabla - (\alpha_0^2 - |v_0|^2)\sigma_3. \tag{2.54}$$

In turn, to $\mathcal{H}(W_0)$ we associate the zero spectrum projection $P_0(W_0)$, the imaginary spectrum projection $P_{im}(W_0) = P_+(W_0) + P_-(W_0)$, and the continuous spectrum projection $P_c(W_0) = I - P_0(W_0) - P_{im}(W_0)$.

Up to quadratic corrections, the centre-stable submanifold is given by the affine subspace

$$W_0 + (P_c(W_0) + P_-(W_0))\dot{H}^{1/2} = \{W_0 + R_0 \mid R_0 \in \dot{H}^{1/2},\ (P_0(W_0) + P_+(W_0))R_0 = 0\}. \tag{2.55}$$

This submanifold will have codimension nine, so we need a supplementary argument, presented at the end, to recover eight codimensions.

Take initial data of the form

$$R(0) = R_0 + hF^+(W_0), \pi(0) = \pi_0 = (\alpha_0, \Gamma_0, (v_k)_0, (D_k)_0), \tag{2.56}$$

where $R_0 \in (P_c + P_-)\dot{H}^{1/2}$. Thus, R_0 belongs to the appropriate linear subspace, while hF^+ is the quadratic correction, made in the direction of F^+; in particular,

$$P_0(0)R(0) = 0,\ P_+R(0) = h. \tag{2.57}$$

(2.56) also implies that $\|R(0)\|_{\dot{H}^{1/2}} \leq C(\|R_0\|_{\dot{H}^{1/2}} + |h|)$.

We define a map Υ as follows:

Definition 2.1. *Υ is the map that, given a pair (R^0, π^0), associates to it the unique bounded solution (R, π) of the linearized equation system (2.33) from Lemma 2.2, with initial data as in (2.56), where R_0 is given and h is allowed to take any value:*

$$\Upsilon((R^0, \pi^0)) = (R, \pi). \tag{2.58}$$

In the sequel we show that, for given R_0, R^0, and π^0, the *bounded* solution (R, π) exists and is unique and that the parameter $h = h(R_0, R^0, \pi^0)$ is determined by the condition that the solution should have finite X seminorm. There exist a unique value of h for which the solution is bounded and a unique solution (R, π) corresponding to that value of h. Thus the map Υ is well-defined.

Consider the space

$$X = \{(R, \pi) \mid R \in L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6}_x,\ \dot{\pi} \in L^1\}, \tag{2.59}$$

with the seminorm

$$\|(R, \pi)\|_X = \|R\|_{L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6}_x} + \alpha_0^{-1}\|\dot{\pi}\|_{L^1_t}, \tag{2.60}$$

where α_0 is the scaling parameter, see (2.56). X is the natural space for the study of equation (0.1) and of its linearized version (2.33). Indeed, since the Schrödinger equations (0.1) and (2.17) are $\dot{H}^{1/2}$-critical, we need to study them in the critical Strichartz space $L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6}_x$. In regard to the modulation path, $\dot{\pi} \in L^1$ is a minimal assumption to ensure we are dealing with small perturbations — and there is no room for a stronger condition, due to working in a critical space.

We prove that, for $\|(R^0, \pi^0)\|_X < \delta$, it follows that $\|(R, \pi)\|_X = \|\Upsilon(R^0, \pi^0)\|_X < \delta$ as well, when δ is small. For clarity, we state the stability result formally:

Proposition 2.3. *Let (R, π) be a solution of (2.33) with initial data given by (2.56). Assume that the auxiliary functions (R^0, π^0) of (2.33) also satisfy (2.56) and, for some sufficiently small $\delta \leq \delta_0$, $\|(R^0, \pi^0)\|_X \leq \delta$. Then $(R, \pi) \in X$ for a unique value of h, which we denote $h(R_0, R^0, \pi^0)$, and this bounded solution satisfies*

$$\begin{aligned} \|R\|_{L_t^\infty \dot{H}_x^{1/2} \cap L_t^2 \dot{W}_x^{1/2,6}} &\leq C(\|R_0\|_{\dot{H}^{1/2}} + \delta^2), \\ \|\dot{\pi}\|_{L_t^1} + |h(R_0, R^0, \pi^0)| &\leq C\alpha_0(\delta\|R_0\|_{\dot{H}^{1/2}} + \delta^2). \end{aligned} \tag{2.61}$$

Supposing that $\|R_0\|_{\dot{H}^{1/2}} \leq c\delta$ and δ is small, it follows that the set

$$\{(R, \pi) \mid R(0) = R_0 + hF^+(W_0),\ \pi(0) = \pi_0,\ \|R, \pi\|_X \leq \delta\} \tag{2.62}$$

is stable under the action of the map Υ and that

$$|h(R_0, R^0, \pi^0)| \leq C\delta^2. \tag{2.63}$$

To a first order, R itself is then given by the time evolution of R_0 under $\mathcal{H}_{\pi^0}(t)$ and all other terms in its composition are of higher order (of size δ^2).

We claim the previous result for each W_0 and sufficiently small δ, but are also interested in the dependence of δ on W_0. Since the Strichartz norms are scaling-invariant, they are left unchanged by symmetry transformations; π and h scale like α_0. After accounting for this fact, we are left with an invariant statement.

Proof. The modulation path $\pi^0 : [0, \infty) \to \mathbb{R}^8$ determines the moving soliton w_{π^0}, following (2.30), and the time-dependent Hamiltonian $\mathcal{H}_{\pi^0}(t)$, as per Lemma 2.2.

Having in view Theorem 3.9 and the preceding discussion in Section 3.3, recall the notations

$$\begin{aligned} w(\pi^0(t)) &= e^{i(\Gamma^0(t) + v^0(t) \cdot x)} \phi(x - D^0(t), \alpha^0(t)), \\ W(\pi^0(t)) &= \begin{pmatrix} w(\pi^0(t)) \\ \overline{w}(\pi^0(t)) \end{pmatrix}, \end{aligned} \tag{2.64}$$

and consider the family of isometries

$$U(t) = e^{\int_0^t (2v^0(s)\nabla + i((\alpha^0(s))^2 - |v^0(s)|^2)\sigma_3)\, ds}. \tag{2.65}$$

Observe that

$$\begin{aligned} W(\pi^0(t)) &= U(t) W_{\pi^0}(t), \\ \mathcal{H}(W(\pi^0(t))) &= U(t)^{-1} \mathcal{H}(W_{\pi^0}(t)) U(t), \end{aligned} \tag{2.66}$$

and let

$$Z(t) = U(t) R(t). \tag{2.67}$$

For brevity we denote

$$\begin{aligned} \mathcal{H}(t) &= \mathcal{H}(W(\pi^0(t))), & P_0(t) &= P_0(W(\pi^0(t))), \\ P_c(t) &= P_c(W(\pi^0(t))), & P_{im}(t) &= P_{im}(W(\pi^0(t))), \\ P_+(t) &= P_+(W(\pi^0(t))), & P_-(t) &= P_-(W(\pi^0(t))), \\ F^+(t) &= F^+(W(\pi^0(t))), & F^-(t) &= F^-(W(\pi^0(t))), \\ \Xi_f(t) &= \Xi_f(W(\pi^0(t))), & \sigma(t) &= \sigma(W(\pi^0(t))). \end{aligned} \tag{2.68}$$

Rewriting the equation (2.33) from Lemma 2.2

$$i\partial_t R - \mathcal{H}_{\pi^0}(t) R = F(t) \tag{2.69}$$

with Z as the unknown, we obtain

$$i\partial_t Z - \mathcal{H}(t)Z = U(t)F(t). \tag{2.70}$$

Instead of fixing a time-independent Hamiltonian for this equation, we consider the time-dependent Hamiltonian $\mathcal{H}(t)$ and divide the equation for Z into three parts, according to the three components of the spectrum — continuous, null, and imaginary. Let

$$I = P_c(t) + P_0(t) + P_{im}(t),\ \ P_{im}(t) \quad = P_+(t) + P_-(t). \tag{2.71}$$

Then, we separately prove, for each of the three components, estimates that enable us to carry out the contraction scheme.

The P_0 component is the most straightforward. By (2.56), the orthogonality condition

$$\langle R(t), \Xi_f(W_{\pi^0}(t))\rangle = 0 \tag{2.72}$$

holds at time $t = 0$ and the modulation equations (2.33) then imply that the orthogonality condition still holds at any other time t. Applying the isometry $U(t)$, the orthogonality condition turns into

$$\langle Z(t), \Xi_f(t)\rangle = 0, \tag{2.73}$$

which directly implies that

$$P_0(t)Z(t) = 0. \tag{2.74}$$

The continuous spectrum projection of Z fulfills the equation, derived from (2.70),

$$i\partial_t(P_c(t)Z) + \mathcal{H}(t)P_c(t)Z = P_c(t)U(t)F(t) + i(\partial_t P_c(t))Z. \tag{2.75}$$

The right-hand side term

$$F = -iL_{\pi^0}R + N(R^0, \pi^0) - N_{\pi^0}(R^0, \pi^0) \tag{2.76}$$

is bounded in the dual Strichartz norm by means of the fractional Leibniz rule:

$$\begin{aligned} \|L_{\pi^0}R\|_{L^2_t \dot W^{1/2,6/5}_x} &\leq C\delta\|R\|_{L^2_t \dot W^{1/2,6}_x} = C\delta\|Z\|_{L^2_t \dot W^{1/2,6}_x}, \\ \|N(R^0, \pi^0) - N_{\pi^0}(R^0, \pi^0)\|_{L^2_t \dot W^{1/2,6/5}_x} &\leq C\delta^2. \end{aligned} \tag{2.77}$$

When using the fractional Leibniz rule, there is an endpoint Sobolev embedding issue, namely that $\dot W^{1/2,6}$ does not embed into L^∞. However, when estimating the quadratic and cubic terms, such as $|r^0|^2 r^0$, present in $N(R^0, \pi^0)$, we can avoid the problem by putting r^0 in non-endpoint Strichartz spaces:

$$\begin{aligned} \|(r^0)^2 w(\pi^0)\|_{L^2_t \dot W^{1/2,6/5}_x} &\leq C\|(r^0)^2\|_{L^2_t \dot H^{1/2}_x} \|w(\pi^0)\|_{L^\infty_t \dot H^{1/2}_x} \\ &\leq C\|R^0\|^2_{L^4_t W^{1/2,3}_x} \|W(\pi^0)\|_{L^\infty_t \dot H^{1/2}_x}. \end{aligned} \tag{2.78}$$

The same applies to the cubic term, except that there we replace $w(\pi^0)$ by r^0.

The endpoint Strichartz space intervenes only when evaluating linear terms such as $L_{\pi^0}R$. However, there we avoid the attendant endpoint Sobolev embedding issue by means of a less sharp fractional Leibniz rule, which we can use because solitons are of Schwartz class.

Furthermore,

$$\begin{aligned}(\partial_t P_+(t))Z &= (\partial_t \alpha^{-3})\langle Z, i\sigma_3 F^-(t)\rangle F^+(t) + \alpha^{-3}\langle Z, i\sigma_3 \partial_t F^-(t)\rangle F^+(t)+\\ &+ \alpha^{-3}\langle Z, i\sigma_3 F^-(t)\rangle \partial_t F^+(t)\\ &= -3\alpha^{-4}\dot\alpha\langle Z, i\sigma_3 F^-(t)\rangle F^+(t) + \alpha^{-3}\langle Z, i\sigma_3 d_\pi F^-(t)\dot\pi^0(t)\rangle F^+(t)+\\ &+ \alpha^{-3}\langle Z, i\sigma_3 F^-(t)\rangle d_\pi F^+(t)\dot\pi^0(t)\end{aligned} \tag{2.79}$$

and likewise for P_- and P_0. Since $P_c = I - P_0 - P_{im}$, by direct examination it follows that

$$\|(\partial_t P_\pm(t))Z\|_{L^2_t \dot W^{1/2,6/5}_x} \leq C\delta\|Z\|_{L^2_t \dot W^{1/2,6}_x}. \tag{2.80}$$

Provided $\|\dot\pi^0\|_1 < \delta$ is sufficiently small, Theorem 3.9 leads to Strichartz estimates for $P_c(t)Z$, by means of the following construction. $\mathcal{H}(t)$ is given by

$$\mathcal{H}(t) = \begin{pmatrix} \Delta + 2|w(\pi^0(t))|^2 & w^2(\pi^0(t)) \\ -\overline{w}^2(\pi^0(t)) & -\Delta - 2|w(\pi^0(t))|^2 \end{pmatrix} + 2iv^0(t)\nabla - ((\alpha^0(t))^2 - |v^0(t)|^2)\sigma_3. \tag{2.81}$$

Denote, for $\pi^0(0) = \pi_0$ following (2.56),

$$\tilde{\mathcal{H}}(t) = \begin{pmatrix} \Delta + 2|w(\pi_0)|^2 & w^2(\pi_0) \\ -\overline{w}^2(\pi_0) & -\Delta - 2|w(\pi_0)|^2 \end{pmatrix} + 2iv^0(t)\nabla - ((\alpha^0(t))^2 - |v^0(t)|^2)\sigma_3. \tag{2.82}$$

The difference $\tilde{\mathcal{H}} - \mathcal{H}$ is small in the appropriate $\dot W^{1/2,6/5-\epsilon} \cap \dot W^{1/2,6/5+\epsilon}$ norm (in much stronger norms too), so the corresponding term can be bounded by means of endpoint Strichartz estimates:

$$\begin{aligned}\|(\tilde{\mathcal{H}} - \mathcal{H})Z\|_{L^2_t \dot W^{1/2,6/5}_x} &\leq \|\tilde{\mathcal{H}}(t) - \mathcal{H}(t)\|_{L^\infty_t \dot W^{1/2,6/5-\epsilon}\cap \dot W^{1/2,6/5+\epsilon}} \|Z\|_{L^2_t \dot W^{1/2,6}_x}\\ &\leq C\|\pi^0(t) - \pi_0\|_{L^\infty_t}\|Z\|_{L^2_t \dot W^{1/2,6}_x}\\ &\leq C\|\dot\pi^0\|_1 \|Z\|_{L^2_t \dot W^{1/2,6}_x}\\ &\leq C\delta\|Z\|_{L^2_t \dot W^{1/2,6}_x}.\end{aligned} \tag{2.83}$$

Then, by (2.77), (2.80), and (2.83), for sufficiently small δ and under the assumption that there are no embedded eigenvalues we can apply Theorem 3.9 with the Hamiltonian $\tilde{\mathcal{H}}$ and obtain

$$\|P_c(t)Z\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} \leq C(\|Z(0)\|_{\dot H^{1/2}} + \delta^2 + \delta\|Z\|_{L^2_t \dot W^{1/2,6}_x}). \tag{2.84}$$

For the imaginary spectrum projection, write, following Section 2.2,

$$P_{im}(t)Z(t) = b^+(t)F^+(t) + b^-(t)F^-(t). \tag{2.85}$$

Then, by taking the time derivative of

$$b^\pm(t) = (\alpha^0(t))^{-3}\langle Z(t), i\sigma_3 F^\mp(t)\rangle, \tag{2.86}$$

we obtain

$$\begin{aligned}\partial_t b^\pm &= (\alpha^0)^{-3}\langle \partial_t Z, i\sigma_3 F^\mp\rangle - 3\dot\alpha^0(\alpha^0)^{-4}\langle Z, i\sigma_3 F^\mp(t)\rangle +\\ &+ (\alpha^0)^{-3}\langle Z, i\sigma_3(d_\pi F^\mp)\dot\pi^0\rangle.\end{aligned} \tag{2.87}$$

Substituting $\partial_t Z$ by its expression given by equation (2.33), we arrive at

$$\begin{aligned}\partial_t b^\pm &= \pm\sigma(t)b^\pm - \langle UF, \sigma_3 F^\mp\rangle -\\ &- 3\dot\alpha^0(\alpha^0)^{-4}\langle Z, i\sigma_3 F^\mp\rangle + (\alpha^0)^{-3}\langle Z, i\sigma_3(d_\pi F^\mp)\dot\pi^0\rangle.\end{aligned} \tag{2.88}$$

Thus b^+ and b^- satisfy the equation

$$\partial_t \begin{pmatrix} b_- \\ b_+ \end{pmatrix} + \begin{pmatrix} \sigma(t) & 0 \\ 0 & -\sigma(t) \end{pmatrix} \begin{pmatrix} b_- \\ b_+ \end{pmatrix} = \begin{pmatrix} N_-(t) \\ N_+(t) \end{pmatrix}, \tag{2.89}$$

where

$$N_\pm = -\langle UF, \sigma_3 F^\mp \rangle - 3\dot{\alpha}^0 (\alpha^0)^{-4} \langle Z, i\sigma_3 F^\mp \rangle + (\alpha^0)^{-3} \langle Z, i\sigma_3 (d_\pi F^\mp) \dot{\pi}^0 \rangle. \tag{2.90}$$

Here $\pm i\sigma(t)$ are the imaginary eigenvalues of $\mathcal{H}(t)$, as in our discussion of its spectrum in Section 2.2.

Concerning the right-hand side, (2.88) and (2.77) imply

$$\|N_\pm(t)\|_{L^2_t} \leq C(\delta \|Z(t)\|_{L^2_t \dot{W}^{1/2,6}_x} + \delta^2). \tag{2.91}$$

To control the solution we use the following simple fact, see [SCH]. It characterizes the bounded solution of the ordinary differential equation system (2.89).

Lemma 2.4. *Consider the equation*

$$\dot{x} - \begin{pmatrix} \sigma(t) & 0 \\ 0 & -\sigma(t) \end{pmatrix} x = f(t), \tag{2.92}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *is the unknown,* $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^1_t \cap L^\infty_t$*, and* $\sigma(t) \geq \sigma_0 > 0$ *is bounded from below. Then* x *is bounded on* $[0, \infty)$ *if and only if*

$$0 = x_1(0) + \int_0^\infty e^{-\int_0^t \sigma(\tau)\, d\tau} f_1(t)\, dt. \tag{2.93}$$

In this case, for all $t \geq 0$

$$\begin{aligned} x_1(t) &= -\int_t^\infty e^{\int_s^t \sigma(\tau)\, d\tau} f_1(s)\, ds, \\ x_2(t) &= e^{-\int_0^t \sigma(\tau)\, d\tau} x_2(0) + \int_0^t e^{-\int_s^t \sigma(\tau)\, d\tau} f_2(s)\, ds. \end{aligned} \tag{2.94}$$

Proof. Any solution will be a linear combination of the exponentially increasing and the exponentially decaying ones and we want to make sure that the exponentially increasing one is absent. It is always true that

$$\begin{aligned} x_1(t) &= e^{\int_0^t \sigma(\tau)\, d\tau} \Big(x_1(0) + \int_0^t e^{-\int_0^s \sigma(\tau)\, d\tau} f_1(s)\, ds \Big), \\ x_2(t) &= e^{-\int_0^t \sigma(\tau)\, d\tau} x_2(0) + \int_0^t e^{-\int_s^t \sigma(\tau)\, d\tau} f_2(s)\, ds. \end{aligned} \tag{2.95}$$

Thus, if x_1 is to remain bounded, the expression between parantheses must converge to 0, hence (2.93). Conversely, if (2.93) holds, then

$$x_1(t) = -\int_t^\infty e^{\int_s^t \sigma(\tau)\, d\tau} f_1(s)\, ds \tag{2.96}$$

tends to 0. □

Note that σ depends Lipschitz continuously on the scaling parameter α, in an explicit manner, following (2.49). Then $\sigma(t)$ belongs to a compact subset $[a_1, a_2]$ of $(0, \infty)$, because $\alpha^0(t)$ belongs to a compact subset of $(0, \infty)$. Consequently, equation (2.89) has a bounded solution if and only if

$$0 = b_+(0) + \int_0^\infty e^{-\int_0^t \sigma(\tau)\, d\tau} N_+(t)\, dt. \tag{2.97}$$

One sees that $b_+(0) = h$, where $b_+(0)$ is given by (2.97) and h by (2.56). R is globally bounded in time if and only if Z is bounded. Z is bounded if and only if each of its components is bounded, $P_{im} Z$ in particular. Thus R is bounded only if

$$h = -\int_0^\infty e^{-\int_0^t \sigma(\tau)\, d\tau} N_+(t)\rangle\, dt. \tag{2.98}$$

We are interested in a more direct formula for h, one that involves R instead of Z. Note that $\sigma(\tau)$ is also the imaginary eigenvalue of $\mathcal{H}(W_{\pi^0}(\tau))$. Expanding N_+ and reverting the isometry U within (2.98) leads to the explicit formula

$$\begin{aligned} h = -\int_0^\infty & e^{-\int_0^t \sigma(W_{\pi^0}(\tau))\, d\tau} \big(\langle F, \sigma_3 F^-(W_{\pi^0}(t))\rangle - \\ & - 3\dot\alpha^0(t)(\alpha^0(t))^{-4} \langle R, i\sigma_3 F^-(W_{\pi^0}(t))\rangle + \\ & + (\alpha^0(t))^{-3} \langle R, i\sigma_3 (d_\pi F^-(W_{\pi^0}(t)))\dot\pi^0(t)\rangle\big)\, dt. \end{aligned} \tag{2.99}$$

It remains to show that, for this unique value of h, R is indeed bounded.

We obtain

$$\begin{aligned} |h| &\leq C \int_0^\infty e^{-\int_0^t \sigma(\tau)\, d\tau} |N_+(t)|\, dt \\ &\leq C \int_0^\infty e^{-ta_1} |N_+(t)|\, dt \\ &\leq C \|N_+\|_{L^1_t + L^\infty_t} \\ &\leq C(\delta \|Z\|_{L^2_t \dot W^{1/2,6}_x} + \delta^2). \end{aligned} \tag{2.100}$$

Following (2.94), both b_+ and b_- are given by convolutions with exponentially decaying kernels in t, whose rate of decay is bounded from below:

$$\begin{aligned} |b_+(t)| &\leq \int_t^\infty e^{(t-s)a_1} |N_+(s)|\, ds, \\ |b_-(t)| &\leq \int_{-\infty}^t e^{-(t-s)a_1} |N_-(s)|\, ds + e^{-ta_1} \|R_0\|_{\dot H^{1/2}}, \end{aligned} \tag{2.101}$$

with the convention that $N(s) = 0$ for $s < 0$; the second term for $b_-(t)$ stems from $e^{-t\sigma} b_-(0)$. One has

$$\begin{aligned} \|P_{im}(t) Z\|_{L^2_t \dot W^{1/2,6}_x} &\leq \|b_+\|_{L^2_t} + \|b_-\|_{L^2_t} \\ &\leq C(\|N_+\|_{L^2_t} + \|N_-\|_{L^2_t} + \|R_0\|_{\dot H^{1/2}}) \\ &\leq C(\|R_0\|_{\dot H^{1/2}} + \delta \|R\|_{L^2_t \dot W^{1/2,6}_x} + \delta^2). \end{aligned} \tag{2.102}$$

Putting all three estimates (2.74), (2.84), and (2.102) together and taking into account the fact that

$$\|Z\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} = \|R\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x}, \tag{2.103}$$

we obtain

$$\|R\|_{L_t^2 \dot{W}_x^{1/2,6}} \leq C(\|R_0\|_{\dot{H}^{1/2}} + \delta \|R\|_{L_t^2 \dot{W}_x^{1/2,6}} + \delta^2). \tag{2.104}$$

Concerning the modulation path π, from the modulation equations (2.33) we get that

$$\|\dot{\pi}\|_1 \leq C(\delta \|Z\|_{L_t^2 \dot{W}_x^{1/2,6}} + \delta^2). \tag{2.105}$$

Overall,

$$\|(R, \pi)\|_X = \|(Z, \pi)\|_X \leq C(\|R_0\|_{\dot{H}_x^{1/2}} + \delta \|(R, \pi)\|_X + \delta^2) \tag{2.106}$$

and this proves stability for the suitable choice of h and small initial data R_0. $\square$

2.4. **The fixed point argument: contraction.** The parameter δ was chosen such that the scaling parameter $\alpha(t)$ belongs to a fixed compact subset of $(0, \infty)$ and therefore the imaginary eigenvalue $i\sigma$ fulfills

$$\sigma \in [a_1, a_2] \subset (0, \infty), \tag{2.107}$$

for all the admissible paths that we consider. Fix, then, a constant $\rho \in (0, a_1)$.

For any two bounded solutions of the linearized equation (2.33), $(R_j, \pi_j) = \Upsilon(R_j^0, \pi_j^0)$, $j = 1, 2$, located in X, such that

$$\|(R_j^0, \pi_j^0)\|_X \leq \delta, \tag{2.108}$$

we prove that Υ acts as a contraction in the following space Y:

$$Y = \{(R, \pi) \mid \|e^{-t\rho} R(t)\|_{L_t^\infty \dot{H}_x^{1/2} \cap L_t^2 \dot{W}_x^{1/2,6}} + \|e^{-t\rho} \dot{\pi}(t)\|_{L_t^1} < \infty\}. \tag{2.109}$$

This is only a seminorm, but defines a metric space for fixed initial data.

Furthermore, for fixed R_0 we prove that the unique value of h, $h(R_0, R^0, \pi^0)$, for which the solution with initial data (2.56) is bounded, satisfies

$$|h(R_0, R_1^0, \pi_1^0) - h(R_0, R_2^0, \pi_2^0)| \leq C\delta \|(R_1^0, \pi_1^0) - (R_2^0, \pi_2^0)\|_Y. \tag{2.110}$$

Observe that this enough to complete the proof. For initial data given by (2.56) and fixed initial data R_0 and π_0, consider a sequence

$$(R_n, \pi_n) = \Upsilon((R_{n-1}, \pi_{n-1})), \ \|(R_n, \pi_n)\|_X \leq \delta \tag{2.111}$$

which converges in the Y sense to (R, π); the parameters $h_n = h(R_0, R_n^0, \pi_n^0)$ form a Cauchy sequence as well. Then the pair (R, π) is a fixed point of Υ and, by virtue of Lemma 2.2, a solution to the nonlinear equation (locally in time in a weak sense and therefore globally as well) with the specified initial data and, furthermore,

$$\|(R, \pi)\|_X \leq \limsup \|(R_n, \pi_n)\|_X \leq \delta. \tag{2.112}$$

This follows first on any finite time interval $[0, T]$ and then in the limit on $[0, \infty)$.

We seek to prove that for any sufficiently small choice of δ

$$\|(R_1, \pi_1) - (R_2, \pi_2)\|_Y \leq 1/2 \|(R_1^0, \pi_1^0) - (R_2^0, \pi_2^0)\|_Y. \tag{2.113}$$

Letting R_1 and R_2 start from distinct initial data proves useful, leading to the formulation of the following perturbation lemma, which we employ repeatedly:

Lemma 2.5. *Consider two solutions, (R_1, π_1) and (R_2, π_2), of two distinct linearized equations in the form (2.33):*

$$
\begin{aligned}
&i\partial_t R_j + \mathcal{H}_{\pi_j^0}(t) R_j = F_j,\ F_j = -iL_{\pi_j^0} R_j + N(R_j^0, W_{\pi_j^0}) - N_{\pi_j^0}(R_j^0, W_{\pi_j^0}) \\
&\dot f_j = 4\alpha_j^0 \|W_{\pi_j^0}\|_2^{-2}(\langle R_j, (d_\pi \Xi_f(W_{\pi_j^0}))\dot\pi_j^0\rangle - i\langle N(R_j^0, W_{\pi_j^0}), \Xi_f(W_{\pi_j^0})\rangle), f \in \{\alpha, \Gamma\} \\
&\dot f_j = 2\|W_{\pi_j^0}\|_2^{-2}(\langle R_j, (d_\pi \Xi_f(W_{\pi_j^0}))\dot\pi_j^0\rangle - i\langle N(R_j^0, W_{\pi_j^0}), \Xi_f(W_{\pi_j^0})\rangle), f \in \{v_k, D_k\}.
\end{aligned}
\tag{2.114}
$$

for $j = \overline{1,2}$, with initial data

$$R_j(0) = R_{0j} + h_j F^+(W(\pi_0)),\ \pi_j(0) = \pi_j^0(0) = \pi_0 \text{ given.} \tag{2.115}$$

Assume in addition that $\|(R_j^0, \pi_j^0)\|_X \leq \delta$ and

$$(R_j, \pi_j) = \Upsilon((R_j^0, \pi_j^0)),\ h_j = h(R_{0j}, R_j^0, \pi_j^0), \tag{2.116}$$

meaning that h_j hold the unique values that make (R_j, π_j) the bounded solutions of (2.114). Then, assuming $\delta > 0$ is sufficiently small,

$$
\begin{aligned}
\|(R_1, \pi_1) - (R_2, \pi_2)\|_Y &\leq C\delta \|(R_1^0, \pi_1^0) - (R_2^0, \pi_2^0)\|_Y + C\|R_{01} - R_{02}\|_{\dot H^{1/2}}, \\
|h_1 - h_2| &\leq C\delta(\|(R_1^0, \pi_1^0) - (R_2^0, \pi_2^0)\|_Y + \|R_{01} - R_{02}\|_{\dot H^{1/2}}).
\end{aligned}
\tag{2.117}
$$

In this formulation, δ and various constants denoted by C may depend on the value of ρ in the definition of Y, (2.109), which in turn depends on the scaling parameter α.

Proof. Let $R = R_1 - R_2$, $\pi = \pi_1 - \pi_2$. R_j, $j = 1, 2$, satisfy the equations

$$i\partial_t R_j + \mathcal{H}_{\pi_j^0}(t) R_j = F_j, \tag{2.118}$$

with initial data

$$R_j(0) = R_{0j} + h_j F^+(W(\pi_0)). \tag{2.119}$$

Note that, by the preceding stability result, Proposition 2.3, $\|(R_j, \pi_j)\|_X \leq \delta$ for $j = 1, 2$. Furthermore, R_j satisfy the orthogonality conditions

$$\langle R_j(t), \Xi_f(W_{\pi_j^0}(t))\rangle = 0 \tag{2.120}$$

at time $t = 0$ and thus, due to equation (2.114), at every time t.

Subtracting the linearized equations from one another, we obtain a similar one for the difference $R = R_1 - R_2$:

$$i\partial_t R + \mathcal{H}_{\pi_1^0}(t) R = \tilde F,\ \tilde F = F_1 - F_2 - (\mathcal{H}_{\pi_1^0}(t) - \mathcal{H}_{\pi_2^0}(t)) R_2. \tag{2.121}$$

We choose the Hamiltonian $\mathcal{H}_{\pi_1^0}(t)$ (the choice of one or two is arbitrary) and apply the same isometry $U(t)$ as described by (2.65-2.68):

$$U(t) = e^{\int_0^t (2v_1^0(s)\nabla + i((\alpha_1^0)^2(s) - |v_1^0(s)|^2)\sigma_3)\, ds}. \tag{2.122}$$

Let

$$Z(t) = U(t) R(t). \tag{2.123}$$

We reintroduce notations similar to (2.74), namely

$$\begin{aligned} \mathcal{H}(t) &= \mathcal{H}(W(\pi_1^0(t))), & P_0(t) &= P_0(W(\pi_1^0(t))), \\ P_c(t) &= P_c(W(\pi_1^0(t))), & P_{im}(t) &= P_{im}(W(\pi_1^0(t))), \\ P_+(t) &= P_+(W(\pi_1^0(t))), & P_-(t) &= P_-(W(\pi_1^0(t))), \\ F^+(t) &= F^+(W(\pi_1^0(t))), & F^-(t) &= F^-(W(\pi_1^0(t))), \\ \Xi_f(t) &= \Xi_f(W(\pi_1^0(t))), & \sigma(t) &= \sigma(W(\pi_1^0(t))). \end{aligned} \tag{2.124}$$

Rewritten for Z, the equation becomes

$$i\partial_t Z + \mathcal{H}(t)Z = U(t)\tilde{F}. \tag{2.125}$$

Then we split Z into three parts, according to the Hamiltonian's spectrum:

$$Z = P_c(t)Z + P_0(t)Z + P_{im}(t)Z. \tag{2.126}$$

We bound each component of Z in the same manner as in the previous section. The main difference is related to $P_0(t)$, which need no longer be zero because the orthogonality condition does not hold.

For the right-hand side $\tilde{F} = F_1 - F_2 - (\mathcal{H}_{\pi_1^0}(t) - \mathcal{H}_{\pi_2^0}(t))R_2$ we estimate the difference term by term, arriving at the following estimate:

$$\|\tilde{F}\|_{e^{t\rho}L_t^2 \dot{W}_x^{1/2,6/5}} \leq C\delta(1 + \rho^{-2})(\|(R,\pi)\|_Y + \|(R_1^0,\pi_1^0) - (R_2^0,\pi_2^0)\|_Y). \tag{2.127}$$

Here we have taken advantage of the exponential weight, as integrating in time preserves the space $e^{t\rho}L_t^p$, at the cost of a factor of ρ^{-1} :

$$\Big\| \int_0^t f(s)\, ds \Big\|_{e^{t\rho}L_t^p} \leq C\rho^{-1}\|f\|_{e^{t\rho}L_t^p}. \tag{2.128}$$

In particular, one has in any Schwartz seminorm S_x, in the space variables, that

$$\|W_{\pi_1^0}(t) - W_{\pi_2^0}(t)\|_{L_t^\infty S_x} \leq C(1+\rho^{-1})(\|(R_1^0,\pi_1^0) - (R_2^0,\pi_2^0)\|_Y \tag{2.129}$$

The initial data are given by

$$Z(0) = (h_1 - h_2)F^+(W(\pi_0)) + R_{01} - R_{02} \tag{2.130}$$

and satisfy the bound

$$\|Z(0)\|_{\dot{H}^{1/2}} \leq C(|h_1 - h_2| + \|R_{01} - R_{02}\|_{\dot{H}^{1/2}}). \tag{2.131}$$

It is straightforward to bound the continuous spectrum projection, $P_c(t)Z$, which satisfies the equation

$$i\partial_t(P_c(t)Z) + \mathcal{H}(t)P_c(t)Z = P_c(t)U(t)\tilde{F} + i(\partial_t P_c(t))Z. \tag{2.132}$$

Starting from the usual Strichartz estimates of Theorem 3.9 and integrating against the exponential weight $e^{t\rho}$, we obtain Strichartz estimates with this weight:

$$\begin{aligned} &\|P_c(t)Z\|_{e^{t\rho}(L_t^\infty \dot{H}_x^{1/2} \cap L_t^2 \dot{W}_x^{1/2,6/5})} \\ \leq& C\Big(\|P_c(0)Z(0)\|_{\dot{H}^{1/2}} + \|\tilde{F}\|_{e^{t\rho}L_t^2\dot{W}_x^{1/2,6}} + \|\partial_t P_c(t))Z\|_{e^{t\rho}L_t^2\dot{W}_x^{1/2,6}}\Big) \\ \leq& C\|R_{01} - R_{02}\|_{\dot{H}^{1/2}} + C\delta(1+\rho^{-1})(\|(R_1^0,\pi_1^0) - (R_2^0,\pi_2^0)\|_Y + \|(R,\pi)\|_Y). \end{aligned} \tag{2.133}$$

There is no contribution due to $h_1 - h_2$, since $P_c(0)Z(0)$ does not depend on $h_1 - h_2$.

R satisfies no orthogonality condition, but R_1 and R_2 do. Taking the difference of these two relations leads to

$$\langle R_1(t) - R_2(t), \Xi_f(W_{\pi_1^0}(t))\rangle = \langle R_2(t), \Xi_f(W_{\pi_2^0}(t)) - \Xi_f(W_{\pi_1^0}(t))\rangle. \tag{2.134}$$

Applying the isometry U, it follows that

$$\langle Z(t), \Xi_f(t)\rangle = \langle R_1(t) - R_2(t), \Xi_f(W_{\pi_1^0}(t))\rangle \tag{2.135}$$

and therefore

$$\|P_0(t)Z(t)\|_{e^{t\rho}(L_t^\infty \dot H_x^{1/2} \cap L_t^2 \dot W_x^{1/2,6/5})} \leq C\delta(1+\rho^{-2})\|(R_1^0, \pi_1^0) - (R_2^0, \pi_2^0)\|_Y. \tag{2.136}$$

Concerning the imaginary component, Lemma 2.4 applies again, since Z is globally bounded forward in time, as seen from

$$\|P_{im}(t)Z(t)\|_{\dot H^{1/2}} \leq \|R_1(t)\|_{\dot H^{1/2}} + \|R_2(t)\|_{\dot H^{1/2}} < C < \infty. \tag{2.137}$$

Write, following Section 2.2,

$$P_{im}(t)Z(t) = b^+(t)F^+(t) + b^-(t)F^-(t). \tag{2.138}$$

Then, as in the proof of the stability result, we infer

$$\begin{aligned} \partial_t b^\pm &= \pm\sigma(t)b^\pm - \langle U\tilde F, \sigma_3 F^\mp\rangle - \\ &\quad - 3\dot\alpha^0(\alpha^0)^{-4}\langle Z, i\sigma_3 F^\mp\rangle + (\alpha^0)^{-3}\langle Z, i\sigma_3(d_\pi F^\mp)\dot\pi^0\rangle. \end{aligned} \tag{2.139}$$

Thus b^+ and b^- satisfy the equation

$$\partial_t \begin{pmatrix} b_- \\ b_+ \end{pmatrix} + \begin{pmatrix} \sigma(t) & 0 \\ 0 & -\sigma(t) \end{pmatrix} \begin{pmatrix} b_- \\ b_+ \end{pmatrix} = \begin{pmatrix} N_-(t) \\ N_+(t) \end{pmatrix}, \tag{2.140}$$

where

$$N_\pm = -\langle U\tilde F, \sigma_3 F^\mp\rangle - 3\dot\alpha_1^0(\alpha_1^0)^{-4}\langle Z, i\sigma_3 F^\mp\rangle + (\alpha_1^0)^{-3}\langle Z, i\sigma_3(d_\pi F^\mp)\dot\pi_1^0\rangle. \tag{2.141}$$

Then,

$$\|N_\pm\|_{e^{t\rho}L_t^2} \leq C\delta\|(R,\pi)\|_Y. \tag{2.142}$$

As a consequence of the lemma, we obtain that $P_{im}(t)Z$ is in the same space as $N_\pm$, as a function of time. Indeed, here we emphasize that since $\rho < a_1 < \sigma(t)$, convolution with $e^{-a_1|t|}$ preserves the space $e^{t\rho}L_t^2$. One gets that

$$\begin{aligned} |h_1 - h_2| = |b_+(0)| &\leq C\int_0^\infty e^{-ta_1}|N_+(t)|\,dt \\ &\leq C\delta(\|(Z_1^0, \pi_1^0) - (Z_2^0, \pi_2^0)\|_Y + \|Z\|_{e^{t\rho}\dot H_x^{1/2}}). \end{aligned} \tag{2.143}$$

Also through Lemma 2.4, it follows that

$$\|P_{im}Z\|_Y \leq C\delta(\|(Z_1^0, \pi_1^0) - (Z_2^0, \pi_2^0)\|_Y + \|(R,\pi)\|_Y). \tag{2.144}$$

Finally, subtracting the modulation equations (2.114), the difference between the paths, $\pi = \pi_1 - \pi_2$, also fulfills

$$\|\dot\pi\|_{e^{t\rho}L_t^1} \leq C\delta(1+\rho^{-2})(\|(Z_1^0, \pi_1^0) - (Z_2^0, \pi_2^0)\|_Y + \|(Z,\pi)\|_Y). \tag{2.145}$$

Up to this point, there was no actual need for exponential weights and we could have used polynomial weights in their stead. However, (2.145) makes exponential weights necessary. The modulation equations involve terms of the form $\dot f$ on the left-hand side and $\int_0^t f(s)\,ds$

on the right-hand side. The only way to contain both within the same space involves exponential weights.

Putting (2.133), (2.136), (2.144), and (2.145) together we retrieve estimate (2.117). □

Remark 2.6. *A difference estimate also exists in the case of different starting solitons, $\pi_j^0(0) = \pi_j(0) = \pi_{0j}$, $j = 1, 2$. In order to perform a comparison, we first make the starting solitons coincide by means of a symmetry transformation and then compare the solutions in a unified setting.*

More precisely, let

$$\mathfrak{g}(t) f(x,t) = e^{i(v\cdot x - t|v|^2 + \Gamma)} \alpha f(\alpha x - 2tv - D, \alpha^2 t) \tag{2.146}$$

be the unique symmetry transformation such that $\mathfrak{g}(0)$ takes $w(\pi_{01})$ to $w(\pi_{02})$. We define accordingly the action of $\mathfrak{g}$ on the modulation parameters:

$$\mathfrak{g}\pi = \tilde\pi \Longleftrightarrow \mathfrak{g} w(\pi) = w(\tilde\pi). \tag{2.147}$$

Then, using the notation $\mathfrak{g}(R, \pi) = (\mathfrak{g}(t)R(t), \mathfrak{g}(t)\pi(t))$, it immediately follows from the previous lemma that

$$\begin{aligned} \|\mathfrak{g}(R_1, \pi_1) - (R_2, \pi_2)\|_Y &\leq C\delta \|\mathfrak{g}(R_1^0, \pi_1^0) - (R_2^0, \pi_2^0)\|_Y + C\|\mathfrak{g}(0)R_{01} - R_{02}\|_{\dot H^{1/2}}, \\ |h_1 - h_2| &\leq C\delta(\|\mathfrak{g}(R_1^0, \pi_1^0) - (R_2^0, \pi_2^0)\|_Y + \|\mathfrak{g}(0)R_{01} - R_{02}\|_{\dot H^{1/2}}). \end{aligned} \tag{2.148}$$

2.5. **Analyticity of the invariant manifold.** Consider a solution of the nonlinear equation (0.1) stemming from the contraction argument presented. For a soliton $W_0 = W(\pi_0)$, it has the form

$$R = W_0 + R_0 + h(R_0, W_0) F^+(W_0). \tag{2.149}$$

Here $F^+(W_0)$ is the eigenvector corresponding to the upper half-plane eigenvalue of $\mathcal{H}(W_0)$ (see Section 2.2), R_0 belongs to the codimension-nine vector space

$$\mathcal{N}_0(W_0) = (P_c(W_0) + P_-(W_0)) \dot H^{1/2}, \tag{2.150}$$

and $h(R_0, W_0)$ is the unique value determined by the contraction argument that leads to an asymptotically stable solution to (0.1), for these initial data.

We recall again that R and other vector-valued functions we are considering have the form $\begin{pmatrix} r \\ \bar r \end{pmatrix}$ and that, as a consequence, the dot product is real-valued.

At this point we give the following formal definition:

Definition 2.2. *Let $\mathcal{M}$ be the eight-dimensional soliton manifold and*

$$\begin{aligned} \mathcal{N}_0(W) &= \{R_0 \in (P_c(W) + P_-(W))\dot H^{1/2} \mid \|R_0\|_{\dot H^{1/2}} < \delta_0\} \\ \mathcal{N}(W) &= \{W + R_0 + h(R_0, W)F^+(W) \mid R_0 \in \mathcal{N}_0(W)\} \\ \mathcal{N}_0 &= \{(R_0, W) \mid R_0 \in \mathcal{N}_0(W),\ W \in \mathcal{M}\} \\ \mathcal{N} &= \bigcup_{W\in\mathcal{M}} \mathcal{N}(W). \end{aligned} \tag{2.151}$$

δ_0 will be chosen independently of W.

The fiber bundle $\mathcal{N}_0$ is trivial over the soliton manifold. Indeed, for each soliton W there exists a unique symmetry transformation $\mathfrak{g}_W$ that takes W_0 (a fixed soliton) into W. Then

$$(R_0, W) \mapsto \mathfrak{g}_W(W_0 + R_0) \tag{2.152}$$

is an isomorphism between a tubular neighborhood of the base in the (trivial) product bundle $(P_c(W_0) + P_-(W_0))\dot{H}^{1/2} \times \mathcal{M}$ (where $\mathcal{M}$ is the soliton manifold) and $\mathcal{N}_0$. This endows $\mathcal{N}_0$ with a real analytic manifold structure.

$\mathcal{N}$ is the image of $\mathcal{N}_0$ under the map

$$\mathcal{F}(R_0, W) = W + R_0 + h(R_0, W)F^+(W). \tag{2.153}$$

Following the contraction argument from beginning to end and giving appropriate values to δ, commensurate with the size of the initial data R_0, we summarize the conclusion as follows (this is the gist of our main theorem):

Proposition 2.7. *There exists $\delta_0 > 0$ such that for each soliton $W_0 = W(\pi_0)$ there is a map $h(\cdot, W_0) : \mathcal{N}_0(W_0) \to \mathbb{R}$ such that*

(1) *h is locally Lipschitz continuous in both variables,*
(2) *$|h(R_0, W_0)| \leq C\alpha_0 \|R_0\|^2_{\dot{H}^{1/2}}$ (α_0 is the scaling parameter of W_0),*

and

$$\mathcal{F}(R_0, W_0) = W_0 + R_0 + h(R_0, W_0)F^+(W_0) \tag{2.154}$$

gives rise to an asymptotically stable solution Ψ to (0.1) with $\Psi(0) = \mathcal{F}(R_0, W_0)$ such that

$$\Psi(t) = W_\pi(t) + R(t). \tag{2.155}$$

Here W_π is a moving soliton with $W_\pi(0) = W_0$, governed in accordance to (1.24) by a path π such that

$$\|\dot{\pi}\|_1 \leq C\alpha_0 \|R_0\|^2_{\dot{H}^{1/2}}. \tag{2.156}$$

R is in the Strichartz space, with initial data $R(0) = R_0 + h(R_0, W_0)F^+(W_0)$, and

$$\|R\|_{L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6}_x} \leq C\delta. \tag{2.157}$$

Adopting a different point of view, for sufficiently small δ_0 $\|R_0\|_{\dot{H}^{1/2}}$ is comparable to the distance from $\Psi(0)$ to the soliton manifold,

$$\min_W \|\Psi(0) - W\|_{\dot{H}^{1/2}}. \tag{2.158}$$

Indeed, one inequality is obvious and the other follows by Lemma 2.12, see below.

Applying the perturbation Lemma 2.5 to the solution of the nonlinear equation leads to the following:

Proposition 2.8. *The solution depends continuously on the initial data: given solutions*

$$\Psi_j(t) = W_{\pi_j}(t) + R_j(t), \tag{2.159}$$

$j = 1, 2$, with initial data in the same fiber, $\Psi_j(0) \in \mathcal{N}(W_0)$, meaning $\Psi_j(0) = \mathcal{F}(R_{0j}, W_0)$, one has

$$\begin{aligned} \|\Psi_1 - \Psi_2\|_{e^{t\rho}(L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6/5}_x)} &\leq C\|(R_1, \pi_1) - (R_2, \pi_2)\|_Y \\ &\leq C\|R_{01} - R_{02}\|_{\dot{H}^{1/2}} \end{aligned} \tag{2.160}$$

and

$$|h(R_{01}, W_0) - h(R_{02}, W_0)| \leq C\alpha_0(\|R_{01}\|_{\dot{H}^{1/2}} + \|R_{02}\|_{\dot{H}^{1/2}})\|R_{01} - R_{02}\|_{\dot{H}^{1/2}}. \tag{2.161}$$

The continuous correspondence between different fibers is given by symmetry transformations.

Proof. Firstly, applying the comparison Lemma 2.5 yields

$$\|(R_1,\pi_1)-(R_2,\pi_2)\|_Y \leq C\|R_{01}-R_{02}\|_{\dot H^{1/2}}. \tag{2.162}$$

The comparison Lemma 2.5 also implies

$$|h(R_{01},W_1)-h(R_{02},W_2)| \leq C\delta\|R_{01}-R_{02}\|_{\dot H^{1/2}}. \tag{2.163}$$

Taking δ proportional to $\|R_{01}\|_{\dot H^{1/2}}+\|R_{02}\|_{\dot H^{1/2}}$ leads to the second conclusion. $\square$

In particular, this shows that the map $\mathcal{F}$ given by (2.153) is locally Lipschitz continuous. We explore its properties further, beginning with a definition and a preliminary lemma.

Definition 2.3. *Given two Banach spaces A and B, a map $f: A \to B$ is analytic if it admits a Taylor series expansion:*

$$f(a) = f_0 + f_1(a) + f_2(a,a) + \ldots, \tag{2.164}$$

such that for each n f_n is n-linear and there exist constants C_1, C_2 such that $\|f_n\|_{A^{\otimes n}\to B} \leq C_1C_2^n$.

A similar definition can be given for differentiable, C^n class, and smooth maps. Furthermore, the definition naturally extends to manifolds.

Lemma 2.9. *The map $\tilde F : \mathcal{N}_0 \times \mathbb{R} \to \dot H^{1/2}$,*

$$\tilde F(W,R,h) = W + R + hF^+(W), \tag{2.165}$$

is locally a real analytic diffeomorphism in the neighborhood of each point $(W,R,0)$.

Proof. Let $W_0 = W(\pi_0)$ and consider the differential of $\tilde F$, given by the linear map $d\tilde F\mid_{(W_0,R_0,h_0)}: \mathbb{R}^8 \times \mathcal{N}_0(W_0) \times \mathbb{R} \to \dot H^{1/2}$,

$$d\tilde F\mid_{(W_0,R_0,h_0)}(\delta\pi,\delta R,\delta h) = (d_\pi W_0)\delta\pi + \delta R + (\delta h)F^+(W_0) + h_0(d_\pi F^+(W_0))\delta\pi. \tag{2.166}$$

That $d\tilde F$ is bijective at points where $h_0=0$ follows from the following identity:

$$\begin{aligned}\Psi &= P_0(W_0)\Psi + P_c(W_0)\Psi + P_-(W_0)\Psi + P_+(W_0)\Psi\\ &= \sum_{f\in\{\alpha,\Gamma,v_k,D_k\}} \langle\Psi,\Xi_f(W_0)\rangle\partial_f W_0 + (P_c(W_0)+P_-(W_0))\Psi +\\ &\quad+\alpha_0^{-3}\langle\Psi, i\sigma_3F^-(W_0)\rangle F^+(W_0).\end{aligned} \tag{2.167}$$

This leads to an explicitly constructed inverse for the linear map: if

$$\Psi = d\tilde F\mid_{(W_0,R_0,h_0)}(\delta\pi,\delta R,\delta h), \tag{2.168}$$

then

$$\begin{aligned}\delta\pi &= (\langle\Psi,\Xi_\alpha(W_0)\rangle, \langle\Psi,\Xi_\Gamma(W_0)\rangle, \langle\Psi,\Xi_{v_k}(W_0)\rangle, \langle\Psi,\Xi_{D_k}(W_0)\rangle),\\ \delta R &= (P_c(W_0)+P_-(W_0))\Psi,\\ \delta h &= \alpha_0^{-3}\langle\Psi, i\sigma_3F^-(W_0)\rangle.\end{aligned} \tag{2.169}$$

The local invertibility of the nonlinear map $\tilde F$ follows by the inverse function theorem. Smoothness follows by inspection of the explicit forms of $\partial_f W_0$ and $\partial_f F^+(W_0)$.

Next, we consider the analyticity of $\tilde{F}$, which is closely tied to that of the soliton $W(\pi)$, of its derivatives, and of the eigenvectors $F^+(W(\pi))$, considered as functions of the parameters π.

The analyticity of the soliton was shown by Li–Bona [LiBo], but we need a stronger statement than the one proved in their paper.

Lemma 2.10. *Given an exponentially decaying solution ϕ of equation (0.3)*

$$-\Delta\phi(\cdot,\alpha)+\alpha^2\phi(\cdot,\alpha)=\phi^3(\cdot,\alpha),$$

the soliton

$$w(\pi)(x)=e^{i(x\cdot v+\Gamma)}\phi(x-D,\alpha) \tag{2.170}$$

is a real analytic function of π in any Schwartz class seminorm S. Furthermore,

$$\|\partial_\alpha^{\beta_\alpha}\partial_\Gamma^{\beta_\Gamma}\partial_{v_k}^{\beta_{v_k}}\partial_{D_k}^{\beta_{D_k}}w\|_S\leq C_1C_2^{|\beta|}\prod_{f\in\{\alpha,v_k\}}\beta_f!. \tag{2.171}$$

We distinguish between two kinds of parameters in the definition of π (2.170). In the parameters α and v_k, the proof makes it clear that the domain of analyticity is not the whole complex plane, but only a strip along the real axis. In the parameters Γ and D_k, on the other hand, the soliton extends to an analytic function of exponential type in the whole complex plane.

Proof. To begin with, we show that w is an analytic $e^{-(1-\epsilon)|x|}L^\infty_x$-valued map. Since the derivatives of an analytic map are also analytic and $\partial_{D_k}w(\pi)=-\partial_{x_k}w(\pi)$, by iterating it follows that arbitrarily many derivatives of $w(\pi)$ are analytic maps into this space.

The real analyticity of w is equivalent to its complex analyticity and the joint complex analyticity is equivalent to separate analyticity in each variable. Given that $\phi(\cdot,\alpha)$ decays exponentially at the rate $e^{-(\alpha^2-\epsilon)|x|}$, by the Agmon bound, we can extend w to an analytic function for all Γ and on the strip $\{v\mid|\operatorname{Im}v|<\alpha^2\}$. This proves analyticity in regard to Γ and v.

Analyticity in the other two variables, D and α, requires that we show that $\phi(\cdot,1)$ is an analytic function in the spatial variables, in the sense that there exist constants C_1 and C_2 such that for every multiindex $\beta=(\beta_{v_1},\beta_{v_2},\beta_{v_3})$

$$\|\partial^\beta\phi\|_2\leq C_1|\beta|!C_2^{|\beta|}. \tag{2.172}$$

ϕ is exponentially decaying, $|\phi(x)|\leq Ce^{-(1-\epsilon)|x|}$, and satisfies the equation

$$-\Delta\phi+\phi=g(\phi), \tag{2.173}$$

where $g(z)=z^3$ is an analytic function. Differentiating β times, we obtain

$$\begin{aligned}(-\Delta+1)(\partial^\beta\phi)&=\partial^\beta(g\circ\phi)\\&=\sum_{j=1}^{|\beta|}\sum_{\beta_1+\ldots+\beta_j=\beta}\partial^{\beta_1}\phi\cdot\ldots\cdot\partial^{\beta_j}\phi\cdot(g^{(j)}\circ\phi).\end{aligned} \tag{2.174}$$

In the proof we make the more general assumption that g is analytic and its derivatives grow subexponentially, meaning that there for any $C_4>0$ there exists C_3 such that

$$|g^{(j)}(0)|\leq C_3C_4^j. \tag{2.175}$$

It follows that for any $C_4 > 0$ there exists C_3 such that, for fixed $f \in H^2$,

$$\|g^{(j)} \circ f\|_{H^2} \leq C_3 C_4^j. \tag{2.176}$$

This property characterizes not only polynomials such as the case of interest, $g(z) = z^3$, but also functions of subexponential growth, such as $g(z) = \cosh(\sqrt{z})$.

To avoid complications, we present the argument in the algebra H^2. We prove by induction on $|\beta|$ that, for properly chosen C_1 and C_2,

$$\|\partial^\beta \phi\|_{H^2} \leq \frac{C_1 C_2^{|\beta|}}{|\beta|^2 + 1}. \tag{2.177}$$

Indeed, assuming that this induction hypothesis holds for all indices up to $|\beta|$, we obtain that

$$\begin{aligned} \|\partial^\beta \phi\|_{H^4} &\leq C\|(-\Delta + 1)(\partial^\beta \phi)\|_{H^2} \\ &\leq C C_3 C_2^{|\beta|} \sum_{j=1}^{|\beta|} \sum_{\beta_1 + \ldots + \beta_j = \beta} \frac{C_4^j C_1^j}{(|\beta_1|^2 + 1) \cdot \ldots \cdot (|\beta_j|^2 + 1)}. \end{aligned} \tag{2.178}$$

Observe that

$$\sum_{k=0}^{n} \frac{1}{(k^2+1)((n-k)^2+1)} < \frac{6}{n^2+1}. \tag{2.179}$$

By induction, we obtain that

$$\sum_{n_1 + \ldots + n_j = n} \frac{1}{(n_1^2 + 1) \cdot \ldots \cdot (n_j^2 + 1)} \leq \frac{6^{j-1}}{n^2 + 1}. \tag{2.180}$$

Therefore

$$\|\partial^\beta \phi\|_{H^4} \leq \frac{C C_3 C_2^{|\beta|}}{|\beta|^2 + 1} \sum_{j=1}^{|\beta|} (6 C_4 C_1)^j. \tag{2.181}$$

By making C_4 sufficiently small and C_2 sufficiently large, we obtain that the sum is uniformly bounded, regardless of $|\beta|$, and

$$\|\partial^\beta \phi\|_{H^4} \leq \tilde{C} \frac{C_2^{|\beta|}}{|\beta|^2 + 1} \leq \frac{C_1 C_2^{|\beta|+1}}{|\beta|^2 + 1}. \tag{2.182}$$

This proves that the induction assumption also holds for derivatives of order $|\beta| + 1$.

The preceding proof shows that $\phi(\cdot - D)$ is an analytic H^2-valued map. However, the argument works for any algebra A containing ϕ and with the property that

$$\|df\|_A \leq C\|(-\Delta + 1)f\|_A. \tag{2.183}$$

This means that the algebra A has to be invariant under convolution with the kernels

$$\frac{x_k e^{-|x|}}{|x|^3} + \frac{x_k e^{-|x|}}{|x|^2}. \tag{2.184}$$

Note that the algebra $A = e^{-(1-\epsilon)|x|} L^\infty$ fulfills both requirements. Thus, $\phi(\cdot - D)$ is an A-valued analytic map. This ensures the joint analyticity of $w(\pi)$ given by (2.170) in the variables D, v, and Γ around the point $\phi = w(1, 0, 0, 0)$. By symmetry transformations, this implies analyticity around any other point.

Concerning α, observe that the derivative with respect to α is given by the generator of dilations, which is a combination of multiplication by x and taking the gradient ∇:

$$\partial_\alpha \phi = \alpha^{-1}\phi + x\nabla\phi. \tag{2.185}$$

Therefore, analyticity in α follows from that with respect to to D and v.

If W is analytic, so are its derivatives up to any finite order, enabling us to conclude that W is analytic in any Schwartz class seminorm. $\square$

In a similar manner, the analyticity of $F^+(W(\pi))$ as a function of π reduces to that of a fixed eigenfunction F^+. $F^+ = \begin{pmatrix} f^+ \\ \overline{f}^+ \end{pmatrix}$ (see Section 2.2) satisfies the equation

$$(\Delta - 1 + 2\phi^2)f^+ + \phi^2\overline{f}^+ = i\sigma f^+. \tag{2.186}$$

Knowing that ϕ is analytic, proving the analyticity of f^+ proceeds in exactly the same manner as above.

Clearly, if $W(\pi)$ is analytic then so are its derivatives up to any finite order. The same goes for $F^+(W(\pi))$ and its derivatives. This immediately implies that $\tilde{F}$ is real analytic. $\square$

Lemma 2.9 has the following immediate consequence:

Proposition 2.11. *$\mathcal{F}$ given by (2.153) is locally one-to-one and its inverse (defined on its range) is locally Lipschitz.*

Proof. The local invertibility of $\mathcal{F}$ follows immediately from the previous lemma. Indeed, one has

$$\mathcal{F}(R, W) = \tilde{F}(W, R, h(R, W)). \tag{2.187}$$

For a sufficiently small δ_0, $h(R, W)$ is close to zero and the previous lemma applies. In order to establish the Lipschitz property for the inverse, we can simply ignore the parameter h. $\square$

Another consequence is that, if a function is sufficiently close to the manifold $\mathcal{M}$ of solitons, we can project it on the manifold as follows.

Lemma 2.12. *For every soliton W there exists $\delta > 0$ such that whenever $\|\Psi - W\|_{\dot{H}^{1/2}} < \delta$ there exists W_1 such that $P_0(W_1)(\Psi - W_1) = 0$ and*

$$\|\Psi - W_1\|_{\dot{H}^{1/2}} \leq C\|\Psi - W\|_{\dot{H}^{1/2}}. \tag{2.188}$$

Furthermore, W_1 depends Lipschitz continuously on Ψ.

Again, following the use of symmetry transformations, δ can be chosen without regard for W.

Proof. If δ is sufficiently small, $\Psi = \tilde{F}(W_1, R, h)$ for some (W_1, R, h) close to $(W, 0, 0)$ by Lemma 2.10. Since the inverse of $\tilde{F}$ is bounded,

$$\|W - W_1\|_{\dot{H}^{1/2}} + \|R\|_{\dot{H}^{1/2}} + |h| \leq C\|\Psi - W\|_{\dot{H}^{1/2}} \leq C\delta. \tag{2.189}$$

Since $R \in \mathcal{N}_0(W_1)$, by definition $P_0(W_1)(\Psi - W_1) = 0$.

To a first order $W - W_1$ lies in a direction tangent to the soliton manifold $\mathcal{M}$, meaning within the range of $P_0(W_1)$, so by means of a Taylor expansion we obtain

$$\|(I - P_0(W_1))(W - W_1)\|_{\dot{H}^{1/2}} \leq C\|W - W_1\|^2_{\dot{H}^{1/2}}. \tag{2.190}$$

Thus

$$\begin{aligned}\|\Psi - W_1\|_{\dot H^{1/2}} &= \|(I - P_0(W_1))(\Psi - W_1)\|_{\dot H^{1/2}} \\ &\leq C\|\Psi - W\|_{\dot H^{1/2}} + \|(I - P_0(W_1))(W - W_1)\|_{\dot H^{1/2}} \\ &\leq C(\|\Psi - W\|_{\dot H^{1/2}} + \|W - W_1\|^2_{\dot H^{1/2}}).\end{aligned} \tag{2.191}$$

On the other hand,

$$\begin{aligned}\|W - W_1\|^2_{\dot H^{1/2}} &\leq (\|\Psi - W\|_{\dot H^{1/2}} + \|\Psi - W_1\|_{\dot H^{1/2}})^2 \\ &\leq C\|\Psi - W\|_{\dot H^{1/2}} + C\delta\|\Psi - W_1\|_{\dot H^{1/2}}.\end{aligned} \tag{2.192}$$

For δ sufficiently small, the conclusion follows. □

This lemma would be superfluous if P_0 were an orthogonal projection and the constant could be taken to be one then. Also note that in this generality the conclusion still holds for $\dot W^{1/2,6}$, with the same proof.

Definition 2.4. *By small asymptotically stable solution we mean one that can be written as $\Psi(t) = W_\pi(t) + R(t)$ where $W_\pi(t)$ is a moving soliton governed by the parameter path π as in (*1.24*) and*

$$\|(R, \pi)\|_X = \|R\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} + \|\dot\pi\|_1 < \delta_0. \tag{2.193}$$

X is the space that appears in the contraction argument, see (2.59).

We can rewrite any small asymptotically stable solution Ψ as $W_{\tilde\pi}(t) + \tilde R(t)$ such that the orthogonality condition is satisfied:

$$P_0(W(\tilde\pi(t)))\tilde R(t) = 0. \tag{2.194}$$

Following Lemma 2.12, $\tilde R(t)$ is still small in the space $L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x$.

Furthermore, $W_{\tilde\pi}(t)$ depends Lipschitz continuously on Ψ. Writing the modulation equations explicitly as in (2.21), it follows that $\|\dot{\tilde\pi}\|_{L^1_t}$ is small too.

Thus, it makes no difference whether we assume the orthogonality condition initially, as part of Definition 2.4, since we can produce it in this manner. We arrive at the same definition with or without the orthogonality condition, though it may be for different values of δ_0.

Clearly, every solution with initial data on the manifold $\mathcal N$ is small and asymptotically stable. A partial converse is also true.

Proposition 2.13. *If $\Psi(0)$ is the initial value of a small asymptotically stable solution Ψ in the sense of Definition* 2.4 *to equation (*0.1*), then $\Psi(0) \in \mathcal N$.*

Proof. Write $\Psi = W_\pi(t) + R(t)$, satisfying the orthogonality condition

$$P_0(W_\pi(t))R(t) = 0. \tag{2.195}$$

By Lemma 2.9, there exist $R_0 \in \mathcal N_0(W_\pi(0))$ and h such that

$$R(0) = R_0 + h_0 F^+(W_\pi(0)). \tag{2.196}$$

Then, we note that both the initial data

$$\Psi(0) = W_\pi(0) + R_0 + h_0 F^+(W_\pi(0)) \tag{2.197}$$

and the initial data

$$\tilde{\Psi}(0) = W_\pi(0) + R_0 + h(R_0, W_\pi(0))F^+(W_\pi(0)) \tag{2.198}$$

give rise to small asymptotically stable solutions, in the form

$$\Psi(t) = W_\pi(t) + R(t),\ \tilde{\Psi}(t) = W_{\tilde{\pi}}(t) + \tilde{R}(t). \tag{2.199}$$

The perturbation Lemma 2.5 then applies, implying that

$$\|(R, \pi) - (\tilde{R}, \tilde{\pi})\|_Y \leq C\delta \|(R, \pi) - (\tilde{R}, \tilde{\pi})\|_Y. \tag{2.200}$$

Otherwise put, we obtain that $(R, \pi) = (\tilde{R}, \tilde{\pi})$. Applying the lemma once more, it follows that the h values coincide as well, that is $h_0 = h(R_0, W_\pi(0))$. Therefore

$$\Psi(0) = \tilde{\Psi}(0) = W_\pi(0) + R_0 + h(R_0, W_\pi(0))F^+(W_\pi(0)) \tag{2.201}$$

and thus $\Psi(0)$ belongs to $\mathcal{N}$. □

Corollary 2.14. *If Ψ is a solution to (0.1) whose initial data $\Psi(0)$ belongs to $\mathcal{N}$, then $\Psi(t)$ also belongs to $\mathcal{N}$ for all positive t and for sufficiently small negative t.*

Proof. Clearly, both for positive t and for sufficiently small negative t $\Psi(t)$ exists (due to the local existence theory, for negative t) and gives rise to a small asymptotically stable solution. Then, the previous proposition shows that $\Psi(t)$ must still be on the manifold. □

To recapitulate, we have investigated the properties of five maps that describe the solution of the nonlinear problem (0.1), namely

$$h(R_0, W_0) : \mathcal{N}_0 \to \mathbb{R}, \tag{2.202}$$

$$\mathcal{F}(R_0, W_0) : \mathcal{N}_0 \to \mathcal{N},\ \mathcal{F}(R_0, W_0) = \tilde{F}(W_0, R_0, h(R_0, W_0)), \tag{2.203}$$

and the solution itself,

$$\Psi = \Psi(R_0, W_0) : \mathcal{N}_0 \to L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6/5}_x,\ \Psi(t) = W_\pi(t) + R(t), \tag{2.204}$$

$$\pi = \pi(R_0, W_0) : \mathcal{N}_0 \to \dot{W}^{1,1}_t, \tag{2.205}$$

$$R = R(R_0, W_0) : \mathcal{N}_0 \to L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6/5}_x. \tag{2.206}$$

It turns out that all five are real analytic and one can also replace $\mathcal{N}_0$ by $\mathcal{N}$ — change the variable to $\Psi(0)$, that is.

Proposition 2.15. *For fixed W_0 there exists $\rho > 0$ such that the maps h, $\mathcal{F}$ considered as a map into $e^{t\rho} L^\infty_t \dot{H}^{1/2}_x$, (R, π) regarded as a map into Y (see (2.109)), and Ψ, seen as map into $e^{t\rho}(L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6}_x)$, are real analytic in the variable R_0 on the fibre $\mathcal{N}_0(W_0)$ or, equally, in the variable $\Psi(0)$ on the fibre $\mathcal{N}(W_0)$.*

Letting W_0 vary, note that the dependence of these maps on W_0 (across the different fibers of $\mathcal{N}_0$) is given by symmetry transformations, which are also analytic in a properly considered setting.

Also note that the analytic dependence of the solution on initial data and the analyticity of the manifold are closely tied to that of the nonlinearity in (0.1). If the nonlinearity were of class C^k, we would expect that the manifold and the dependence on initial data would also be of class C^k.

Proof. Firstly note that, since $\tilde{F}$ is analytic, it suffices to show that h and (R, π) are analytic.

The proof consists in the following steps: firstly we exhibit the first-order differential of h and (R, π) and show that it satisfies the definition of differentiability, meaning that the remainder in the Taylor expansion is quadratic in size. By recursion, we then introduce the n-th order differentials for arbitrary n.

Finally, we show that differentials and remainders in the Taylor expansion grow in norm at most exponentially, enabling us to conclude that h and (R, π) equal the sum of their own Taylor series within a sufficiently small convergence radius.

To begin with, for given initial data $W_0 = W(\pi_0)$ and R_0, denote the corresponding solution to (0.1) by $\Psi = W_\pi + R$. R and π satisfy the equation system, in the form (2.33),

$$\begin{aligned} &i\partial_t R + \mathcal{H}_\pi(t) R = F(\pi, R), \\ &\dot{f} = F_f(\pi, R),\ f \in \{\alpha, \Gamma, v_k, D_k\}, \end{aligned} \tag{2.207}$$

where

$$\begin{aligned} \mathcal{H}_\pi(t) &= \Delta\sigma_3 + V_\pi(t), \\ F(\pi, R) &= -iL_\pi R + N(R, W_\pi) - N_\pi(R, W_\pi), \end{aligned} \tag{2.208}$$

and we introduced the notation

$$F_f(\pi, R) = \begin{cases} 4\alpha\|W_\pi\|_2^{-2}(\langle R, (d_\pi \Xi_f(W_\pi))\dot{\pi}\rangle \\ \qquad\qquad - i\langle N(R, W_\pi), \Xi_f(W_\pi)\rangle),\ f \in \{\alpha, \Gamma\} \\ 2\|W_\pi\|_2^{-2}(\langle R, (d_\pi \Xi_f(W_\pi))\dot{\pi}\rangle \\ \qquad\qquad - i\langle N(R, W_\pi), \Xi_f(W_\pi)\rangle),\ f \in \{v_k, D_k\}. \end{cases} \tag{2.209}$$

The initial data are given by

$$R(0) = R_0 + h(R_0, W_0)F^+(W_0),\ \pi(0) = \pi_0. \tag{2.210}$$

This setup makes the orthogonality condition

$$\langle R(t), \Xi_f(W_\pi(t))\rangle = 0 \tag{2.211}$$

valid for all times. Indeed, it holds initially at time $t = 0$ and equation (2.207) ensures that it still holds at any other time. We aim to find an infinite expansion of the form

$$R = R^0 + R^1 + R^2 + \ldots,\ \pi = \pi^0 + \pi^1 + \pi^2 + \ldots,\ h(R_0, W_0) = h^0 + h^1 + h^2 + \ldots, \tag{2.212}$$

where $R^n = n! d^n R$, $\pi^n = n! d^n \pi$, and $h^n = n! d^n h$ are the n-th order terms in the power expansions of R, π, and h, given by n-linear expressions in R_0, such that

$$\begin{aligned} \|R - R^0 - R^1 - \ldots - R^n\|_{e^{t\rho}(L_t^\infty \dot{H}_x^{1/2} \cap L_t^2 \dot{W}_x^{1/2,6})} &\leq C_n \|R_0\|_{\dot{H}^{1/2}}^{n+1}, \\ \|\dot{\pi} - \dot{\pi}^0 - \dot{\pi}^1 - \ldots - \dot{\pi}^n\|_{e^{t\rho}L_t^1} &\leq C_n \|R_0\|_{\dot{H}^{1/2}}^{n+1}, \\ |h - h^0 - h^1 - \ldots - h^n| &\leq C_n \|R_0\|_{\dot{H}^{1/2}}^{n+1}. \end{aligned} \tag{2.213}$$

Showing, in addition, that $C_n \leq C_1 C_2^n$ for some constants C_1, C_2 and every n guarantees the analyticity of R, π, and h.

For technical reasons, it is easier to construct the differentials and prove analyticity at zero. Analyticity will then follow at every point within the radius of convergence.

The constant terms in the expansion are

$$R^0(t) = 0,\ \pi^0(t) = \pi_0 = (1, 0, 0, 0),\ h^0 = 0. \tag{2.214}$$

The natural guess is that first-order differentials R^1 and π^1 satisfy the following linearized version of (2.207):

$$\begin{aligned} &i\partial_t R^1 + \mathcal{H}_{\pi^0}(t)R^1 = (\partial_\pi F(\pi^0, R^0))\pi^1 + (\partial_R F(\pi^0, R^0))R^1 - (\partial_\pi V_{\pi^0})(t)\pi^1 R^0 \\ &\dot{f}^1 = (\partial_\pi F_f(\pi^0, R^0))\pi^1 + (\partial_R F_f(\pi^0, R^0))R^1,\ f \in \{\alpha, \Gamma, v_k, D_k\}, \end{aligned} \tag{2.215}$$

with initial data

$$R^1(0) = R_0 + h^1 F^+(W_0),\ \pi^1(0) = 0. \tag{2.216}$$

It is important to note that, because $R^0 = 0$, $\dot\pi^0 = 0$, the following terms cancel:

$$\begin{aligned} &(\partial_\pi V_{\pi^0})(t)\pi^1 R^0 = 0,\ \partial_\pi F_f(\pi^0, R^0) = 0,\ \partial_R F_f(\pi^0, R^0) = 0, \\ &\partial_\pi F(\pi^0, R^0) = 0,\ \partial_R F(\pi^0, R^0) = 0. \end{aligned} \tag{2.217}$$

This improvement only holds when taking the differential at zero and is due to the fact that the nonlinearity is of order higher than two.

It follows that $\pi^1 = 0$ and

$$i\partial_t R^1 + \mathcal{H}_{\pi^0}(t)R^1 = 0. \tag{2.218}$$

The orthogonality condition

$$\langle R, \Xi_f(W_\pi)\rangle = 0 \tag{2.219}$$

becomes

$$\langle R^1, \Xi_f(W_{\pi^0})\rangle + \langle R^0, (\partial_\pi \Xi_f(W_{\pi^0}))\pi^1\rangle = 0; \tag{2.220}$$

consequently it still holds for R^1:

$$\langle R^1, \Xi_f(W_{\pi^0})\rangle = 0. \tag{2.221}$$

By means of Strichartz estimates we obtain, in the same manner as in the proof of stability, that R^1 is bounded for a unique value of h^1 and then it satisfies

$$\|R^1\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} \leq C\|R_0\|_{\dot H^{1/2}}. \tag{2.222}$$

Note that $\|R^1\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x}$ can be made arbitrarily small, but $\|\dot\pi^1\|_{L^1_t} = 0$.

To show that R^1 thus constructed and $\pi^1 = 0$ are indeed the first-order differentials of R and π, consider

$$S^1 = R^0 + R^1, \Sigma^1 = \pi^0 + \pi^1. \tag{2.223}$$

If $F(\pi, R)$, V_π, and $F_f(\pi, R)$ were differentiable, it should be the case that

$$\begin{aligned} F(\Sigma^1, S^1) &= F(R^0, \pi^0) + (\partial_\pi F(\pi^0, R^0))\pi^1 + (\partial_R F(\pi^0, R^0))R^1 + o_1(R^1, \pi^1), \\ V_{\Sigma^1} &= V_{\pi^0} + \partial_\pi V_{\pi^0}\pi^1 + o_2(\pi^1), \\ F_f(\Sigma^1, S^1) &= F_f(\pi^0, R^0) + (\partial_\pi F_f(\pi^0, R^0))\pi^1 + (\partial_R F_f(\pi^0, R^0))R^1 + o_3(R^1, \pi^1). \end{aligned} \tag{2.224}$$

In fact, even more is true: all three quantities are analytic, as one sees by examining their explicit forms.

The main issue is that the soliton $W_\pi(t)$ depends not only on the values of π and $\dot\pi$ at time t, but also on the integral $\int_0^t \pi(s)\,ds$. Thus, even though W_π depends analytically on π, each derivative produces a factor of t.

In this setting, the error terms in (2.224) are quadratic:

$$\begin{aligned}\|o_1(R^1,\pi^1)\|_{\langle t\rangle L^2_t W^{1/2,6/5}_x} &\leq C\|(R^1,\pi^1)\|_X^2 \\ \|o_2(\pi^1)\|_{\langle t\rangle L^\infty_t(\dot W^{1/2,6/5-\epsilon}\cap \dot W^{1/2,6/5+\epsilon})} &\leq C\|(R^1,\pi^1)\|_X^2 \\ \|o_3(R^1,\pi^1)\|_{\langle t\rangle L^1_t} &\leq C\|(R^1,\pi^1)\|_X^2.\end{aligned} \tag{2.225}$$

Therefore

$$\begin{aligned}&i\partial_t S^1+\mathcal{H}_{\Sigma^1}(t)S^1=F(\Sigma^1,S^1)+O^2(R^1,\pi^1)\\ &\dot f_{\Sigma^1}=F_f(\Sigma^1,S^1)+O^2_f(R^1,\pi^1),\ f\in\{\alpha,\Gamma,v_k,D_k\},\end{aligned} \tag{2.226}$$

where O^2, O^2_f are error terms bounded by

$$\|O^2(R^1,\pi^1)\|_{\langle t\rangle L^2_t W^{1/2,6/5}_x}+\|O^2_f(R^1,\pi^1)\|_{\langle t\rangle L^1_t}\leq C\|(R^1,\pi^1)\|^2_X. \tag{2.227}$$

Following (2.221), S^1 fulfills the orthogonality relation

$$\langle S^1(t),\Xi_f(W_{\Sigma^1}(t))\rangle=0. \tag{2.228}$$

Comparing the equation system (2.226) satisfied by S^1 and Σ^1 with the one satisfied by R and π, (2.207), we obtain that

$$\|(R,\pi)-(S^1,\Sigma^1)\|_Y\leq C\|(R^1,\pi^1)\|^2_X\leq C\|R_0\|^2_{\dot H^{1/2}}. \tag{2.229}$$

The proof exactly follows that of Lemma 2.5.

We repeat this procedure for the higher-order terms in the expansion. Let $n\geq 1$ and consider a variation $\pi=\pi^0+\delta\pi$ of π^0 such that $\delta\pi(0)=0$. Recall that solitons depend analytically on parameters: for any Schwartz seminorm S,

$$\|d^n_\pi W(\pi^0)(\delta\pi))\|_S\leq C_1C_2^n n!\|\delta\dot\pi\|_1^n. \tag{2.230}$$

However, by (2.171) only derivatives in α and v_k produce a factorial contribution:

$$\|\partial_\alpha^{\beta_\alpha}\partial_\Gamma^{\beta_\Gamma}\partial_{v_k}^{\beta_{v_k}}\partial_{D_k}^{\beta_{D_k}}w\|_S\leq C_1C_2^{|\beta|}\prod_{f\in\{\alpha,v_k\}}\beta_f!.$$

Thus we obtain for W_{π^0} given by (1.24) and (2.30)

$$\begin{aligned}&W_{\pi^0}(t)=\begin{pmatrix}w_{\pi^0}(t)\\ \overline{w}_{\pi^0}(t)\end{pmatrix},\\ &w_{\pi^0}(t)=w\bigg(\alpha^0(t),\Gamma^0(t)+\int_0^t((\alpha^0(s))^2-|v^0(s)|^2)\,ds,v^0(t),D(t)+2\int_0^t v^0(s)\,ds\bigg)\end{aligned}$$

— and likewise for $\partial_f W_{\pi^0}$, $F^+(W_{\pi^0})$, and all other quantities that depend on the moving soliton — the following explicit expression of analyticity:

$$\|\partial_\alpha^{\beta_\alpha}\partial_\Gamma^{\beta_\Gamma}\partial_{v_k}^{\beta_{v_k}}\partial_{D_k}^{D_k}W_{\pi^0}(t)(\delta\pi))\|_S\leq C_1C_2^{|\beta|}\prod_{f\in\{\alpha,v_k\}}\beta_f!\prod_{f\in\{\Gamma,D_k\}}\langle t\rangle^{\beta_f}\|\delta\dot\pi\|_1^{|\beta|}. \tag{2.231}$$

As noted before, this is an improvement over mere analyticity (and one that is necessary in the sequel), in that, while in half the variables W is only analytic on a strip, in the other half W extends to an analytic function of exponential type on the whole complex plane.

At our discretion, we pick parameters a and a_1 such that $a < a_1 \leq \sigma(\alpha(t))$, for any time t, where $\alpha(t)$ is the scaling component of $\pi(t)$ and π is any path that appears in this proof (their scaling parameters are uniformly bounded away from zero). Define the weights

$$A_n(t) = \sum_{j=0}^{n} \frac{\langle at\rangle^j}{j!} < e^{\langle at\rangle}. \tag{2.232}$$

Then (2.231) implies

$$W_{\pi^0} = W^0_{\pi^0} + W^1_{\pi^0} + \ldots \tag{2.233}$$

such that in any Schwartz seminorm S_x, in the space variables only,

$$\|W^n_{\pi^0}(\delta\pi)\|_{A_n(t)L^\infty_t S_x} \leq C_1 C_2^n \|\delta\dot\pi\|_1^n. \tag{2.234}$$

This immediately yields a power series expansion for V_{π^0}:

$$V_{\pi^0} = V^0_{\pi^0} + V^1_{\pi^0} + \ldots, \tag{2.235}$$

with $V^n_{\pi^0}$ n-linear for each n and

$$\|V^n_{\pi^0}(\delta\pi)\|_{A_n(t)L^\infty_t S_x} \leq C_1 C_2^n \|\delta\dot\pi\|_1^n. \tag{2.236}$$

The weights $A_n(t)$ given by (2.232) have the property that

$$\sum_{j=0}^{n} A_j(t) A_{n-j}(t) \leq C^n A_n(t). \tag{2.237}$$

Consequently, consider the spaces $\partial_t^{-1} A_n(t) L^1_t$. In this setting, for

$$m_1 + \ldots + m_n = m, \tag{2.238}$$

and given n path variations $\delta\pi_1$ to $\delta\pi_n$ such that $\delta\pi_j = 0$, $j = \overline{1,n}$, one has that

$$\|V^n(\delta\pi_1, \ldots, \delta\pi_n)\|_{A_{m+n}(t)L^\infty_t S_x} \leq C_1 C_2^n \|\delta\dot\pi_1\|_{A_{m_1}(t)L^1_t} \cdot \ldots \cdot \|\delta\dot\pi_n\|_{A_{m_n}(t)L^1_t}. \tag{2.239}$$

Considering the explicit form of $F(R,\pi)$ and $F_f(R,\pi)$, note again that

$$F(R^0,\pi^0) = dF(R^0,\pi^0) = 0,\ F_f(R^0,\pi^0) = dF_f(R^0,\pi^0) = 0. \tag{2.240}$$

This is due to linearizing around zero ($R^0 = 0$, $\dot\pi^0 = 0$), starting from a nonlinearity of order higher than two.

In fact, one needs to take two derivatives in R or $\dot\pi$ before arriving at a nonzero derivative of F or F_f. We obtain the power series expansion

$$F = F^0 + F^1 + \ldots,\ F_f = F^0_f + F^1_f + \ldots, \tag{2.241}$$

where F^n and F^n_f are n-linear and, for X defined by (2.59),

$$\begin{aligned} \|F^n\|_{A_{n-2}(t)L^2_t \dot W^{1/2,6/5}_x} &\leq C_1 C_2^n \|(\delta R_1, \delta\pi_1)\|_X \cdot \ldots \cdot \|(\delta R_n, \delta\pi_n)\|_X, \\ \|F^n_f\|_{A_{n-2}(t)L^1_t} &\leq C_1 C_2^n \|(\delta R_1, \delta\pi_1)\|_X \cdot \ldots \cdot \|(\delta R_n, \delta\pi_n)\|_X. \end{aligned} \tag{2.242}$$

The gain from n to $n-2$ is due to the fact that the order of the derivative in π is two less than the total order.

In the scale of weighted spaces

$$X^n = \{(R,\pi) \mid R \in A_n(t)(L^\infty_t \dot H^{1/2}_x \cap L^2_t W^{1/2,6}_x), \dot\pi \in A_n(t)L^1_t\} \tag{2.243}$$

and with the notation

$$m_1 + \ldots + m_n = m, \tag{2.244}$$

one also has that

$$\begin{aligned}\|F^n\|_{A_{m+n-2}(t)L_t^2\dot W_x^{1/2,6/5}} &\leq C_1C_2^n\|(\delta R_1,\delta\pi_1)\|_{X^{m_1}}\cdot\ldots\cdot\|(\delta R_n,\delta\pi_n)\|_{X^{m_n}}\\ \|F^n\|_{A_{m+n-2}(t)L_t^1} &\leq C_1C_2^n\|(\delta R_1,\delta\pi_1)\|_{X^{m_1}}\cdot\ldots\cdot\|(\delta R_n,\delta\pi_n)\|_{X^{m_n}}.\end{aligned} \tag{2.245}$$

Then, in the Taylor expansion of $F(\pi, R)$, where $\pi = \pi^0+\pi^1+\ldots$ and $R = R^0+R^1+\ldots$ are functions of R_0, the n-th order term is

$$\sum_{j=1}^{n}\sum_{n_1+\ldots+n_j=n} F^j((R^{n_1},\pi^{n_1}),\ldots,(R^{n_j},\pi^{n_j})). \tag{2.246}$$

By writing only the n-th order terms in the Taylor expansion of (2.207), we see that R^n and π^n fulfill the equation

$$\begin{aligned}i\partial_t R^n+\mathcal{H}_{\pi^0}(t)R^n &= \sum_{j=2}^{n}\sum_{n_1+\ldots+n_j=n} F^j((R^{n_1},\pi^{n_1}),\ldots,(R^{n_j},\pi^{n_j}))\\ &\quad-\sum_{j=1}^{n}\sum_{n_1+\ldots+n_j+\tilde n=n}(\partial_\pi^j V_{\pi^0})(t)(\pi^{n_1},\ldots,\pi^{n_j})R^{\tilde n}\\ \dot f^n &= \sum_{j=2}^{n}\sum_{n_1+\ldots+n_j=n} F_f^j((R^{n_1},\pi^{n_1}),\ldots,(R^{n_j},\pi^{n_j})).\end{aligned} \tag{2.247}$$

with initial conditions $R^n(0) = h^nF^+(W_0)$ and $\pi^n(0) = 0$.

Since the expansion is around zero, note again the cancellations of all terms containing R^0 or π^0:

$$F^1\mid_{(\pi^0,R^0)}=0,\ F_f^1\mid_{(\pi^0,R^0)}=0,\ (\partial_\pi V_{\pi^0})(t)\pi^nR^0=0. \tag{2.248}$$

This means that highest-order terms, depending on R^n and π^n, are completely absent from the right-hand side, which is consequently given by combinations of lower-order terms.

We assume that the *induction hypothesis*

$$\|(R^m,\pi^m)\|_{X^{m-1}}\leq k_1k_2^m\|R_0\|^m_{\dot H^{1/2}} \tag{2.249}$$

holds for every $m < n$ and some constants k_1 and k_2 to be established later. Then the right-hand sides of equations (2.247) obey the bounds

$$\begin{aligned}&\Big\|\sum_{j=2}^{n}\sum_{n_1+\ldots+n_j=n} F^j((R^{n_1},\pi^{n_1}),\ldots,(R^{n_j},\pi^{n_j}))\Big\|_{A_{n-1}(t)L_t^2\dot W_x^{1/2,6/5}}\leq\\ &\qquad\leq C_1\sum_{j=2}^{n}(C_2k_1)^jk_2^n\|R_0\|^n_{\dot H^{1/2}}\end{aligned} \tag{2.250}$$

and likewise for the other two terms, that is

$$\begin{aligned}&\Big\|\sum_{j=1}^{n}\sum_{n_1+\ldots+n_j+\tilde n=n}(\partial_\pi^j V_{\pi^0})(t)(\pi^{n_1},\ldots,\pi^{n_j})R^{\tilde n}\Big\|_{A_{n-1}(t)L_t^2\dot W_x^{1/2,6/5}}\leq\\ &\qquad\leq C_1\sum_{j=2}^{n}(C_2k_1)^jk_2^n\|R_0\|^n_{\dot H^{1/2}},\end{aligned} \tag{2.251}$$

respectively

$$\Big\| \sum_{j=2}^n \sum_{n_1+\ldots+n_j=n} F_f^j((R^{n_1}, \pi^{n_1}), \ldots, (R^{n_j}, \pi^{n_j})) \Big\|_{A_{n-1}(t) L_t^1} \leq \\ \leq C_1 \sum_{j=2}^n (C_2 k_1)^j k_2^n \|R_0\|^n_{\dot H^{1/2}}. \tag{2.252}$$

Importantly, summation starts from $j = 2$ on the right-hand side, because the terms involving first-order differentials vanish, following (2.248).

The powers of t add up in correctly, since

$$j - 2 + \sum_{j=2}^n (n_j - 1) = n - 2 < n - 1. \tag{2.253}$$

By making k_1 sufficiently small, we can get the sum $\sum_{j=2}^n (C_2 k_1)^j$ to be bounded and small without regard to n and thus we can put

$$\tilde C k_1^2 k_2^n \|R_0\|^n_{\dot H^{1/2}} \tag{2.254}$$

on the right-hand side of (2.250–2.252).

At this point we solve the equation system (2.247) for R^n and π^n in the — by now — customary manner. For the modulation path π^n, (2.252) is exactly the estimate we need.

We split R^n into its projections on the three parts of the spectrum, absolutely continuous, null, and imaginary, and estimate each separately in the weighted Strichartz norm.

As previously (see (2.65-2.68)), we first apply a unitary transformation $U(t)$ to the equation, such that

$$\begin{aligned} U(t) &= e^{\int_0^t (2v^0(s)\nabla + i((\alpha^0)^2(s) - |v^0(s)|^2)\sigma_3)\, ds} \\ Z(t) &= U(t) R^n(t) \\ W(\pi^0) &= U(t) W_{\pi^0}. \end{aligned} \tag{2.255}$$

In case of (2.247), this transformation takes a particularly simple form, as α^0 and v^0 are constant.

For the projection on the continuous spectrum $P_c(t)Z(t)$, Strichartz estimates lead directly to

$$\|P_c(t)Z(t)\|_{A_n(t)(L_t^\infty \dot H_x^{1/2} \cap L_t^2 \dot W_x^{1/2,6})} \leq \tilde C k_1^2 k_2^n \|R_0\|^n_{\dot H^{1/2}}. \tag{2.256}$$

Concerning the projection on the imaginary spectrum, note the convolution estimate

$$\begin{aligned} &\int_0^t e^{-a_1(t-s)} A_n(s)\, ds \leq C A_n(t), \\ &\int_t^\infty e^{a_1(t-s)} A_n(s)\, ds = \int_0^\infty e^{-a_1 s} A_n(t+s)\, ds \leq C A_n(t). \end{aligned} \tag{2.257}$$

Since the right-hand side manifests only polynomial growth, it follows that there exists a unique subexponential solution to the ordinary differential equation that describes the projection on the imaginary spectrum, corresponding to a suitable value of the parameter h^n.

Then we obtain

$$\|P_{im}(t)Z(t)\|_{A_n(t)(L_t^\infty \dot H_x^{1/2} \cap L_t^2 \dot W_x^{1/2,6})} \leq \tilde C k_1^2 k_2^n \|R_0\|^n_{\dot H^{1/2}}. \tag{2.258}$$

The orthogonality condition is fulfilled approximately, in the sense that

$$\langle R^n, \Xi_f(W_{\pi^0})\rangle + \sum_{j=1}^n \sum_{n_1+\ldots+n_j+\tilde n=n} \langle R^{\tilde n}, (\partial_\pi^j \Xi_f(W_{\pi^0}))(\pi^{n_1}, \ldots, \pi^{n_j})\rangle = 0. \tag{2.259}$$

Under our induction hypothesis, this is enough for an appropriate bound on the P_0 component:

$$\|P_0(t)Z(t)\|_{A_n(t)(L_t^\infty \dot H_x^{1/2} \cap L_t^2 \dot W_x^{1/2,6})} \leq \tilde C k_1^2 k_2^n \|R_0\|^n_{\dot H^{1/2}}. \tag{2.260}$$

In conclusion, for the unique suitable value of the parameter h^n,

$$\|(R^n, \pi^n)\|_{X^{n-1}} \leq \tilde C k_1^2 k_2^n \|R_0\|^n_{\dot H^{1/2}}. \tag{2.261}$$

By setting k_1 sufficiently small, we obtain

$$\|(R^n, \pi^n)\|_{X^{n-1}} \leq k_1 k_2^n \|R_0\|^n_{\dot H^{1/2}}. \tag{2.262}$$

In order to compensate for the smallness of k_1, we also need to set k_2 to be large, so that the initial condition for induction

$$\|(R^1, \pi^1)\|_{X^0} \leq k_1 k_2 \|R_0\|_{\dot H^{1/2}} \tag{2.263}$$

is verified.

Next, we verify that R^n and π^n are indeed the n-th order terms in the power series expansion of R and π. Denote

$$S^n = R^0 + R^1 + \ldots + R^n,\ \Sigma^n = \pi^0 + \pi^1 + \ldots + \pi^n. \tag{2.264}$$

We obtain that, up to an error O^{n+1} of order $n+1$, S^n and Σ^n solve an equation system of the form (2.226):

$$\begin{aligned} &i\partial_t S^n + \mathcal{H}_{\Sigma^n}(t) S^n = F(\Sigma^n, S^n) + O^{n+1} \\ &\dot f_{\Sigma^n} = F_f(\Sigma^n, S^n) + O_f^{n+1},\ f \in \{\alpha, \Gamma, v_k, D_k\}, \end{aligned} \tag{2.265}$$

with error terms of size

$$\|O^{n+1}\|_{\langle t\rangle^N L_t^2 W_x^{1/2,6/5}} + \|O_f^{n+1}\|_{\langle t\rangle^N L_t^1} \leq C\|R_0\|^{n+1}_{\dot H^{1/2}}. \tag{2.266}$$

Following a comparison in the exponentially weighted space Y between (S^n, Σ^n) and (R, π), we obtain that

$$\|(R, \pi) - (S^n, \Sigma^n)\|_Y \leq C(n)\|R_0\|^{n+1}_{\dot H^{1/2}}. \tag{2.267}$$

This concludes the proof of analyticity for R and π.

Finally, recall that, by (2.99), $h(R_0, W_0)$ has the formula

$$\begin{aligned} h(R_0, W_0) = -\int_0^\infty & e^{-\int_0^t \sigma(W_\pi(\tau))\, d\tau} \big(\langle F, \sigma_3 F^-(W_\pi(t))\rangle - \\ & - 3\dot\alpha(t)(\alpha(t))^{-4}\langle R, i\sigma_3 F^-(W_\pi(t))\rangle + \\ & + (\alpha(t))^{-3}\langle R, i\sigma_3(d_\pi F^-(W_\pi(t)))\dot\pi(t)\rangle\big)\, dt. \end{aligned} \tag{2.268}$$

Since all the components that enter this formula are analytic and grow in time more slowly than e^{ita_1}, we directly obtain a power series expansion for h.

Alternatively, we can write $h = h^0 + h^1 + \ldots$ and find each term h^n as the initial data of (R^n, π^n) in equation (2.247). h^n is the unique value that makes the solution bounded. By (2.268), it follows that for each n h^n is given by an n-linear form in R_0, whose norm grows only exponentially with n.

Following a comparison between (S^n, Σ^n) and (R, π), we obtain as a byproduct an estimate for the difference

$$|h - (h^0 + \ldots + h^n)| \leq C_1 C_2^n \|R_0\|_{\dot H^{1/2}}^{n+1}. \tag{2.269}$$

This gives us another way to establish the analyticity of h. □

2.6. **The centre-stable manifold.** Finally, we state the connection between the manifold $\mathcal{N}$ issued by our proof and the centre-stable manifold of [BaJo].

Proposition 2.16. *$\mathcal{N}$ is a centre-stable manifold in the sense of Bates–Jones.*

Proof. To begin with, we rewrite equation (0.1) to make it fit the framework of the theory of Bates–Jones [BaJo].

Consider a soliton $W_{\pi_0}(t)$, described by the constant path $\pi_0(t) = (1, 0, 0, 0)$ (without loss of generality). Thus

$$W_{\pi_0}(t) = \begin{pmatrix} e^{it}\phi(\cdot, 1) \\ e^{-it}\phi(\cdot, 1) \end{pmatrix} \tag{2.270}$$

for all t. Linearizing the equation around this constant path yields, for $R = \Psi - W_{\pi_0}$, equation (2.17), which in this case takes the form

$$\partial_t R - i\mathcal{H}_{\pi_0} R = N(R, W_{\pi_0}). \tag{2.271}$$

Making the substitution

$$Z = e^{-it\sigma_3} R, \tag{2.272}$$

we see that

$$i\partial_t Z + \mathcal{H}(W(\pi_0))Z = N(Z, W(\pi_0)), \tag{2.273}$$

where

$$\mathcal{H}(W(\pi_0)) = \mathcal{H} = \begin{pmatrix} \Delta + 2\phi^2(\cdot, 1) - 1 & \phi^2(\cdot, 1) \\ -\phi^2(\cdot, 1) & -\Delta - 2\phi^2(\cdot, 1) + 1 \end{pmatrix} \tag{2.274}$$

and

$$N(Z, W(\pi_0)) = \begin{pmatrix} -|z|^2 z - z^2\phi(\cdot, 1) - 2|z|^2\phi(\cdot, 1) \\ |z|^2 z + z^2\phi(\cdot, 1) + 2|z|^2\phi(\cdot, 1) \end{pmatrix}. \tag{2.275}$$

Note that the right-hand side terms are at least quadratic in Z, due to linearizing around a constant path.

The spectrum of $\mathcal{H}$ is $\sigma(\mathcal{H}) = (-\infty, -1] \cup [1, \infty) \cup \{0, \pm i\sigma\}$; for a more detailed discussion see Section 2.2. The stable spectrum is $-i\sigma$, the unstable spectrum is $i\sigma$, and everything else belongs to the centre.

One can check that all the conditions of [BaJo] are met in regard to (2.273), leading to the existence of a centre-stable manifold in this setting. This was done by [GJLS], whose main theorem we cited in the introduction.

Indeed, in the Banach space $X = H^s$, $s > 3/2$, $\mathcal{H}$ is a closed, densely defined operator with the required spectral properties. The nonlinearity $N(Z_1, W(\pi_0))$ has the Lipschitz property with arbitrarily small constant, since $H^s \subset L^\infty$ is an algebra:

$$\|N(Z_1, W(\pi_0)) - N(Z_2, W(\pi_0))\|_{H^s} \leq C \max(\|Z_1\|_{H^s}, \|Z_2\|_{H^s}) \|Z_1 - Z_2\|_{H^s}. \tag{2.276}$$

We have also exhibited another stable manifold, $\mathcal{N} \subset \dot{H}^{1/2}$ given by Definition 2.2, which is invariant under symmetry transformations and under the time evolution induced by (0.1). Our claim, which we prove in the sequel, is that Ψ belongs to this manifold if and only if Z belongs to the other one, that is

$$\tilde{\mathcal{N}} = \mathcal{N} - \Phi,\ \Phi = \begin{pmatrix} \phi(\cdot, 1) \\ \phi(\cdot, 1) \end{pmatrix} \tag{2.277}$$

is a centre-stable manifold for (2.273) relative to a neighborhood $\mathcal{V}$ of 0, namely $\mathcal{V} = \{Z \mid \|Z\|_{\dot{H}^{1/2}} < \delta_0\}$ for some small δ_0.

The main difference between the two results is that $\dot{H}^{1/2}$ is not an algebra, so the conditions for Bates–Jones's result, Theorem 1.4, are not met in this space. Indeed, if $\psi \in \dot{H}^{1/2}$, it does not follow that $|\psi|^2\psi \in \dot{H}^{1/2}$, much less that this nonlinearity is Lipschitz continuous (see Condition H3 preceding Theorem 1.4).

However, even though the hypothesis fails in $\dot{H}^{1/2}$, the conclusion of [BaJo] — the existence of a centre-stable manifold $\tilde{\mathcal{N}}$ — still holds, following a proper use of Strichartz inequalities.

To show this, we verify the three properties listed in Definition 1.1: $\tilde{\mathcal{N}}$ is t-invariant with respect to a neighborhood of Φ, $\pi^{cs}(\tilde{\mathcal{N}})$ contains a neighborhood of 0 in $X^c \oplus X^s$, and $\tilde{\mathcal{N}} \cap W^u = \{0\}$.

The t-invariance of $\tilde{\mathcal{N}}$ relative to $\mathcal{V}$ follows from definition (see the introduction) and Proposition 2.13. Indeed, the invariance established by Corollary 2.14 is strictly stronger than t-invariance, as it holds globally in time.

The fact that $\pi^{cs}(\tilde{\mathcal{N}})$ contains a neighborhood of 0 in $X^c \oplus X^s$ is a consequence of the local invertibility of $\mathcal{F}$ established in Proposition 2.11.

Finally, we show that $\tilde{\mathcal{N}} \cap W^u = \{0\}$, where W^u is the unstable manifold of the equation (the set of solutions that decay exponentially at $-\infty$). To this purpose, in the sequel we follow the proof of [Bec1] with slight modifications.

Consider a solution $Z^0 \in \tilde{N} \cap W^u$ of (2.273), meaning that Z^0 is defined for all $t < 0$,

$$\|Z^0(t)\|_{\dot{H}^{1/2}} < \delta_0 \tag{2.278}$$

for some small δ_0 and all negative t, and Z^0 decays exponentially as $t \to -\infty$, meaning that for some constants C_1 and C_2 and $t \leq 0$

$$\|Z^0(t)\|_{\dot{H}^{1/2}} \leq C_1 e^{C_2 t} \tag{2.279}$$

(even though polynomial decay is sufficient for a contradiction). Using Strichartz estimates for the free Schrödinger evolution, we obtain that for every T

$$\|Z^0\|_{L^2_t(T,T+1)\dot{W}^{1/2,6}_x} \leq C\|Z^0\|_{L^\infty_t(T,T+1)\dot{H}^{1/2}_x}. \tag{2.280}$$

Therefore, for $T \leq 0$,

$$\|Z^0\|_{L^2_t(-\infty,T]\dot{W}^{1/2,6}_x} \leq Ce^{C_2 T}. \tag{2.281}$$

Thus, $\Psi(t) = e^{it\sigma_3}Z^0(t) + W_{\pi_0}(t)$ is a small asymptotically stable solution of (0.1) in a sense similar to Definition 2.4, but as t goes to $-\infty$.

Therefore, for $t \leq 0$ one can write $\Psi = R(t) + W_\pi(t)$, such that the orthogonality condition

$$P_0(W_\pi(t))R(t) = 0 \tag{2.282}$$

is satisfied and for $T \leq 0$

$$\|R(T)\|_{\dot H^{1/2}} \leq C\delta,\ \|R\|_{L^\infty_t(-\infty,T]\dot H^{1/2}_x \cap L^2_t(-\infty,T]\dot W^{1/2,6}_x} \leq Ce^{C_2 T}. \tag{2.283}$$

The path π satisfies the corresponding modulation equations and

$$\|\dot\pi\|_{L^1_t(-\infty,T]} \leq Ce^{C_2 T}. \tag{2.284}$$

Since $\Psi(0) \in \mathcal{N}$, the path π extends to all positive times and R extends to a solution bounded in the Strichartz norm:

$$\|R\|_{L^\infty_t[0,\infty)\dot H^{1/2}_x \cap L^2_t[0,\infty)\dot W^{1/2,6}_x} \leq C\delta. \tag{2.285}$$

As in previous arguments, let $U(t)$ be the family of isometries defined for π

$$U(t) = e^{\int_0^t (2v(s)\nabla + i(\alpha^2(s) - |v(s)|^2)\sigma_3)\, ds} \tag{2.286}$$

and set

$$Z(t) = U(t)R(t). \tag{2.287}$$

Then $Z(t)$ satisfies the equation

$$i\partial_t Z - \mathcal{H}(t)Z = F. \tag{2.288}$$

Decompose Z into its projections on the continuous, imaginary, and zero spectrum of $\mathcal{H}(t)$ and let

$$\delta(T) = \|Z\|_{L^\infty_t(-\infty,T]\dot H^{1/2}_x \cap L^2_t(-\infty,T]\dot W^{1/2,6}_x} + \|\dot\pi\|_{L^1_t(-\infty,T]}. \tag{2.289}$$

Observe that $\delta(t) \to 0$ as $t \to -\infty$, so we can assume it to be arbitrarily small.

By means of Strichartz estimates one obtains that

$$\begin{aligned}\|P_c(t)Z\|_{L^2_t(-\infty,T]\dot W^{1/2,6}_x \cap L^\infty_t(-\infty,T]\dot H^{1/2}_x} &\leq C\|F\|_{L^2_t(-\infty,T]\dot W^{1/2,6/5}_x + L^1_t(-\infty,T]\dot H^{1/2}_x} \\ &\leq C\delta(T)\|Z\|_{L^\infty_t(-\infty,T]\dot H^{1/2}_x},\end{aligned} \tag{2.290}$$

because the right-hand side contains only quadratic or higher degree terms.

Let

$$P_{im}Z(t) = b_-(t)F^-(t) + b_+(t)F^+(t). \tag{2.291}$$

Since $Z(t)$ is bounded as $t \to -\infty$, we can use Lemma 2.4 for $t \to -\infty$ and obtain

$$\begin{aligned} b_-(t) &= -\int_{-\infty}^t e^{-\sigma(t-s)} N_-(s)\, ds \\ b_+(t) &= e^{(t-T)\sigma} b_+(T) - \int_t^T e^{(t-s)\sigma} N_+(s)\, ds. \end{aligned} \tag{2.292}$$

Therefore

$$\|P_{im}Z\|_{L^\infty_t(-\infty,T]\dot H^{1/2}_x} \leq C(\|P_+Z(T)\|_{\dot H^{1/2}_x} + \delta(T)\|Z\|_{L^\infty_t(-\infty,T]\dot H^{1/2}_x}). \tag{2.293}$$

We have constructed Z such that the orthogonality condition holds, so

$$P_0(t)Z(t) = 0. \tag{2.294}$$

Putting these estimates together, we arrive at

$$\|Z\|_{L^\infty_t(-\infty,T]\dot H^{1/2}_x} \leq C(\delta(T)\|Z\|_{L^\infty_t(-\infty,T]\dot H^{1/2}_x} + \|P_+Z(T)\|_{\dot H^{1/2}}). \tag{2.295}$$

For sufficiently negative T_0, it follows that $\|Z(t)\|_{\dot H^{1/2}} \leq C\|P_+Z(t)\|_{\dot H^{1/2}}$, for any $t \leq T_0$. The converse is obviously true, so the two norms are comparable.

Furthermore, by reiteration one has that

$$\|(I - P_+)Z(t)\|_{\dot H^{1/2}} \leq C\delta(t)\|P_+Z(t)\|_{\dot H^{1/2}}. \tag{2.296}$$

Next, assume that $Z(0)$ is on $\tilde{\mathcal{N}}$, meaning that $Z(0) + \Phi \in \mathcal{N}$.

If the size δ_0 that appears in Definition 2.4 of $\mathcal{N}$ is sufficiently small, it follows that $\|Z(t)\|_{\dot H^{1/2}}$ is bounded from below as $t \to \infty$. Indeed, to a first order, $Z(t)$ is given by the time evolution of the initial data, following (2.218), and this expression cannot vanish.

On the other hand, Lemma 2.4 implies that

$$\|P_+Z(t)\|_{\dot H^{1/2}} \leq \int_t^\infty e^{(t-s)\sigma}|N_+(s)|\, ds \tag{2.297}$$

and thus $\|P_+Z(t)\|_{\dot H^{1/2}}$ goes to zero and can be made arbitrarily small as $t \to \infty$.

Lemma 2.4 of [BaJo] implies, under even more general conditions, that if the ratio $\|P_+Z(T_0)\|_{\dot H^{1/2}}/\|(I - P_+)Z(T_0)\|_{\dot H^{1/2}}$ is small enough, it will stay bounded for all $t \leq T_0$. The proof of this result is based on Gronwall's inequality.

However, this contradicts our previous conclusion (2.296), stating that

$$\|(I - P_+)Z(t)\|_{\dot H^{1/2}}/\|P_+Z(t)\|_{\dot H^{1/2}} \leq C\delta(t) \tag{2.298}$$

goes to 0 as t goes to $-\infty$. Therefore, Z can only be 0.

This proves that $\tilde{\mathcal{N}} \cap W^u = \{0\}$. In other words, there are no exponentially unstable solutions in $\tilde{\mathcal{N}}$ in the sense of [BaJo]. The final requirement for $\tilde{\mathcal{N}}$ to be a centre-stable manifold in the sense of Definition 1.1 is thus met. □

2.7. **Scattering.** Strichartz space bounds and estimates imply that the radiation term scatters like the solution of the free equation, meaning

$$r(t) = e^{-it\Delta}r_{free} + o_{\dot H^{1/2}}(1) \tag{2.299}$$

for some $r_{free} \in \dot H^{1/2}$.

As a reminder, R satisfies the equation (2.17)

$$i\partial_t R - \mathcal{H}_\pi(t)R = F, \tag{2.300}$$

where $\mathcal{H}_\pi(t) = \Delta\sigma_3 + V_\pi(t)$, and R has finite Strichartz norm,

$$\|R\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} < \infty, \tag{2.301}$$

while F has finite dual Strichartz norm,

$$\|F\|_{L^2_t \dot W^{1/2,6/5}_x} < \infty. \tag{2.302}$$

Rewrite (2.300) as

$$i\partial_t R - \Delta\sigma_3 R = F - V_\pi(t)R. \tag{2.303}$$

By Duhamel's formula,

$$R(t) = e^{-it\Delta\sigma_3}R(0) - i\int_0^t e^{-i(t-s)\Delta\sigma_3}(F(s) - V_\pi(s)R(s))\, ds. \tag{2.304}$$

Let

$$R_{free} = R(0) - i\int_0^\infty e^{it\Delta\sigma_3}(F(t) - V_\pi(t)R(t))\, dt. \tag{2.305}$$

Then

$$R(t) - e^{-it\Delta\sigma_3}R_{free} = ie^{-it\Delta\sigma_3}\int_t^\infty e^{is\Delta\sigma_3}(F(s) - V_\pi(s)R(s))\, dt. \tag{2.306}$$

Note that

$$\begin{aligned} &\|F - V_\pi R\|_{L^2_t \dot W^{1/2,6/5}_x} \leq \\ \leq &\|F\|_{L^2_t \dot W^{1/2,6/5}_x} + \|V_\pi\|_{L^\infty_t (\dot W^{1/2,6/5-\epsilon}_x \cap \dot W^{1/2,6/5+\epsilon}_x)} \|R\|_{L^2_t \dot W^{1/2,6}_x} < \infty. \end{aligned} \tag{2.307}$$

implies

$$\lim_{t\to\infty} \|\chi_{[t,\infty)}(s)(F(s) - V_\pi(s)R(s))\|_{L^2_t \dot W^{1/2,6/5}_x} = 0. \tag{2.308}$$

$e^{-it\Delta\sigma_3}$ being an isometry, it follows that $R(t) - e^{-it\Delta\sigma_3} R_{free} \to 0$ in $\dot H^{1/2}$.

This leads to the desired conclusion, upon passing to the scalar functions r and r_{free}, where $R = \begin{pmatrix} r \\ \overline{r} \end{pmatrix}$ and $R_{free} = \begin{pmatrix} r_{free} \\ \overline{r}_{free} \end{pmatrix}$.

3. Linear estimates

We seek a dispersive estimate for the linear time-dependent equation. The highest-order terms we need to take into account are of the form $v(t)\nabla Z(t)$, where $\dot v(t)$ is small in the L^1 norm. Z is a solution of the Schrödinger equation and thus has a finite Strichartz norm.

3.1. Notations.

Consider the linear Schrödinger equation in $\mathbb{R}^3$

$$i\partial_t Z + \mathcal{H} Z = F,\ Z(0) \text{ given}, \tag{3.1}$$

where

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix} + \begin{pmatrix} W_1 & W_2 \\ -W_2 & -W_1 \end{pmatrix}. \tag{3.2}$$

W_1 and W_2 are taken to be real-valued. We also assume that W_1 and W_2 possess a sufficiently large amount of polynomial decay and smoothness, meaning that $\langle x\rangle^{-m}\partial^n V \in L^\infty$ for all m, $n \leq N$ and some sufficiently large N.

All subsequent computations take place under these far from optimal conditions. A derivation of similar results in a sharp (scaling-invariant) or almost sharp setting can be found in [Bec2] and [Bec3].

In keeping with the nonlinear problem, we assume here as well that $\mathcal{H}$ has no eigenvalues or resonances in $(-\infty, -\mu] \cup [\mu, \infty)$.

For simplicity, the entire subsequent discussion revolves around the case of three spatial dimensions. We focus on $\dot H^{1/2}$ estimates, involving half-derivatives, which are of use in the study of the the nonlinear problem under consideration.

Following [Sch] and [ErSc], we list the spectral properties of such operators.

Proposition 3.1. *Let $\mathcal{H} = \mathcal{H}_0 + V$ be as in (3.2) and assume that $\langle x\rangle^{-m}\partial^n V \in L^\infty$ for all m, $n \leq N$ and some sufficiently large N. Then $\sigma(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$ and $\sigma_{ac}(\mathcal{H}) = (-\infty, -\mu] \cup [\mu, \infty)$.*

Assume that $\mathcal{H}$ has no eigenvalues or resonances in $(-\infty, -\mu] \cup [\mu, \infty)$. Then point spectrum is simple, with the possible exception of the eigenvalue at zero. The finitely many Riesz projections P_{ζ_j}, $1 \leq j \leq n$, corresponding to the point spectrum are given by

$$P_{\zeta_j} = \frac{1}{2\pi i} \int_{|z-\zeta_j|=\epsilon} R_V(z)\, dz, \tag{3.3}$$

have finite rank, and their ranges, as well as those of their adjoints $P^*_{\zeta_j}$, are spanned by Schwartz functions. The continuous spectrum projection P_c is given by

$$P_c = I - \sum_{j=1} P_{\zeta_j}. \tag{3.4}$$

We also state the limiting absorption principle for $\mathcal{H}$ as follows.

Proposition 3.2. *Let $\mathcal{H} = \mathcal{H}_0 + V$ be as in (3.2) and assume that $\langle x \rangle^{-m} \partial^n V \in L^\infty$ for all m, $n \leq N$ and some sufficiently large N. Assume that there are no eigenvalues or resonances of $\mathcal{H}$ embedded in $(-\infty, -\mu] \cup [\mu, \infty)$. Then*

$$\sup_{\lambda \in \mathbb{R}} \|R_V(\lambda \pm i0)\|_{L^{6/5} \to L^6} < \infty,\ \sup_{\lambda \in \mathbb{R}} \|R_V(\lambda \pm i0)\|_{\dot W^{1/2,6/5} \to \dot W^{1/2,6}} < \infty. \tag{3.5}$$

Proof. Write $V = V_1 V_2$, where

$$V_1 = \sigma_3 \begin{pmatrix} W_1 & W_2 \\ W_2 & W_1 \end{pmatrix}^{1/2},\ V_2 = \begin{pmatrix} W_1 & W_2 \\ W_2 & W_1 \end{pmatrix}^{1/2},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.6}$$

Since

$$R_V(\lambda) = R_0(\lambda) - R_0(\lambda) V R_0(\lambda) + R_0(\lambda) V_1 (I + V_2 R_0(\lambda) V_1)^{-1} V_2 R_0(\lambda), \tag{3.7}$$

the boundedness of $R_V(\lambda \pm i0)$ follows from the existence of a bounded inverse for $I + V_2 R_0(\lambda \pm i0) V_1$ as an operator from L^2 to itself (or $\dot H^{1/2}$ to itself). By Fredholm's Theorem, this is implied by the nonexistence of a function $f \in L^2$ (or $\dot H^{1/2}$) such that

$$f + V_2 R_0(\lambda \pm i0) V_1 f = 0. \tag{3.8}$$

Indeed, $V_1 f$ would have to be an eigenfunction or resonance for $\mathcal{H}$ at $\lambda \in (-\infty, -\mu] \cup [\mu, \infty)$, contradicting our hypothesis. Note that only $\pm\mu$ could actually be resonances.

Finally, the uniform boundedness of the norm follows from the fact that

$$\lim_{\lambda \to \infty} \|V_2 R_0(\lambda \pm i0) V_1\|_{L^2 \to L^2} = 0 \tag{3.9}$$

and likewise in $\dot H^{1/2}$. □

The resolvent of the unperturbed Hamiltonian, $R_0(\lambda) = (\mathcal{H}_0 - \lambda)^{-1}$, has the kernel

$$R_0(\lambda^2 + \mu)(x, y) = \frac{1}{4\pi} \begin{pmatrix} -\frac{e^{-\sqrt{\lambda^2 + 2\mu}|x-y|}}{|x-y|} & 0 \\ 0 & \frac{e^{i\lambda|x-y|}}{|x-y|} \end{pmatrix}. \tag{3.10}$$

$R_0(\lambda) = (\mathcal{H}_0 - \lambda)^{-1}$ is an analytic function on $\mathbb{C} \setminus ((-\infty, -\mu] \cup [\mu, \infty))$. It can be extended to a continuous function in the closed lower half-plane or the closed upper half-plane, but not both at once, due to a jump discontinuity on the real line.

3.2. **Strichartz estimates.** We begin by recalling the following basic connection between the free evolution $e^{it\mathcal{H}_0}$ and the resolvent.

Lemma 3.3.

$$\|e^{it\mathcal{H}_0}\|_{L^1_t \mathcal{L}(L^1_x \cap L^2_x, L^\infty_x + L^2_x)} < \infty. \tag{3.11}$$

Furthermore, the Fourier transform of $e^{it\mathcal{H}_0}$ is well-defined, in the sense that the integral

$$\lim_{\rho \to \infty} \int_0^\rho e^{-it\lambda} e^{it\mathcal{H}_0}\, dt \tag{3.12}$$

converges, for λ in the lower half-plane, in the $\mathcal{L}(L^1 \cap L^2, L^2 + L^\infty)$ operator norm to $iR_0(\lambda)$ and to $iR_0(\lambda - i0)$ for $\lambda \in \mathbb{R}$. For real λ

$$\lim_{\rho \to \infty} \int_{-\rho}^{\rho} e^{-it\lambda} e^{it\mathcal{H}_0} \, dt = i(R_0(\lambda - i0) - R_0(\lambda + i0)) \tag{3.13}$$

also converges in the operator norm.

Proof. To show (3.11), simply use the dispersive estimate

$$\|e^{it\mathcal{H}_0}\|_{L^1 \to L^\infty} \leq Ct^{-3/2} \tag{3.14}$$

for $t \geq 1$ and the unitarity of the evolution

$$\|e^{it\mathcal{H}_0}\|_{L^2 \to L^2} \leq C \tag{3.15}$$

for $t \leq 1$. Then the integral in (3.12) is well-defined and converges in the operator norm due to dominated convergence. Furthermore,

$$\int_0^\rho e^{-it(\lambda - i\epsilon)} e^{it\mathcal{H}_0} f \, dt = iR_0(\lambda - i\epsilon)(I - e^{-i\rho(\lambda - i\epsilon)} e^{i\rho\mathcal{H}_0}) f. \tag{3.16}$$

Since $f \in L^2$, the right-hand side converges to $iR_0(\lambda - i\epsilon)f$. Letting ϵ go to zero, the result also follows for λ on the real line, by dominated convergence. One must use the explicit form of the kernel (3.10).

Finally, (3.13) follows by separately handling the integrals from $-\rho$ to 0 and from 0 to ρ. □

This admits the following generalization to the case where a potential is present.

Lemma 3.4. *Assume $V \in L^\infty$ and the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V$ is described by (3.2). Then the equation*

$$i\partial_t Z + \mathcal{H} Z = F, \ Z(0) \text{ given}, \tag{3.17}$$

admits a weak solution Z for $Z(0) \in L^2$, $F \in L^\infty_t L^2_x$ and for $t \geq 0$

$$\|Z(t)\|_2 \leq Ce^{t\|V\|_\infty} \|Z(0)\|_2 + \int_0^t e^{(t-s)\|V\|_\infty} \|F(s)\|_2 \, ds. \tag{3.18}$$

Furthermore, $R_V(\lambda)$ is the Fourier transform of $e^{it\mathcal{H}}$: for $\operatorname{Im} \lambda < -\|V\|_\infty$

$$\lim_{\rho \to \infty} \int_0^\rho e^{-it\lambda} e^{it\mathcal{H}} \, dt = iR_V(\lambda). \tag{3.19}$$

Proof. We introduce an auxiliary variable and write

$$i\partial_t Z + \mathcal{H}_0 Z = F - VZ_1, \ Z(0) \text{ given}. \tag{3.20}$$

Over a sufficiently small time interval $[T, T + \epsilon]$, whose size ϵ only depends on $\|V\|_\infty$, the map that associates Z to some given Z_1 is a contraction, in a sufficiently large ball in $L^\infty_t L^2_x$. The fixed point of this contraction mapping is then a solution to (3.17).

This shows that the equation is locally solvable and, by bootstrapping, since the length of the interval is independent of the size of F and of the initial data $Z(T)$, we obtain an exponentially growing global solution. The bound (3.18) follows by Gronwall's inequality.

(3.19) follows in the same manner as (3.12). □

In order to obtain Strichartz estimates in the time-independent case, we first derive the following representation formula (also proved in [Sch] under more restrictive assumptions).

Lemma 3.5. *Let $V \in L^1 \cap L^\infty$ and $\mathcal{H}$ be given by (3.2). Assume that there are no eigenvalues of $\mathcal{H}$ embedded in its essential spectrum $(-\infty, -\mu] \cup [\mu, \infty)$ and only finitely many outside of it. Then for f, $g \in L^2$*

$$\langle P_c f, g\rangle = \frac{1}{2\pi i} \int_{(-\infty,-\mu]\cup[\mu,\infty)} \big\langle (R_V(\lambda - i0) - R_V(\lambda + i0)) f, g\big\rangle \, d\lambda, \tag{3.21}$$

in the sense that the expression under the integral is absolutely integrable and its integral equals the left-hand side.

Proof. Let $V = V_1 V_2$, where

$$V_1 = \sigma_3 \begin{pmatrix} W_1 & W_2 \\ W_2 & W_1 \end{pmatrix}^{1/2}, \ V_2 = \begin{pmatrix} W_1 & W_2 \\ W_2 & W_1 \end{pmatrix}^{1/2}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.22}$$

For $y > \|V\|_\infty$, $I + V_2 R_0(\lambda \pm iy) V_1$ must be invertible. Indeed, V_1 and V_2 are bounded L^2 operators of norm at most $\|V\|_\infty^{1/2}$ and

$$\|R_0(\lambda \pm iy)\|_{2\to 2} \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{y}|x|}}{|x|} \, dx = 1/y. \tag{3.23}$$

Therefore one can explicitly construct the inverse $(I + V_2 R_0(\lambda \pm iy) V_1)^{-1}$ by means of a power series.

By Lemma 3.4 $\chi_{t\geq 0} \langle e^{it\mathcal{H}} e^{-yt} f, g\rangle$ is an exponentially decaying function of t and its Fourier transform is

$$\int_0^\infty \langle e^{-(y+i\lambda)t} e^{it\mathcal{H}} f, g\rangle \, dy = -i \langle R_V(\lambda - iy) f, g\rangle. \tag{3.24}$$

Combining this with the analogous result for the positive side, we see that

$$(\langle e^{it\mathcal{H}} e^{-y|t|} f, g\rangle)^\wedge = i \langle (R_V(\lambda + iy) - R_V(\lambda - iy)) f, g\rangle. \tag{3.25}$$

The right-hand side is absolutely integrable, because

$$R_V(\lambda) = R_0(\lambda) - R_0(\lambda) V R_0(\lambda) + R_0(\lambda) V_1 (I + V_2 R_0(\lambda) V_1)^{-1} V_2 R_0(\lambda) \tag{3.26}$$

and

$$\begin{aligned}
&\int_{-\infty}^\infty |\langle (R_0(\lambda + iy) - R_0(\lambda - iy)) f, g\rangle| \, d\lambda \\
&\leq \int_{-\infty}^\infty \frac{1}{2i} (\langle (R_0(\lambda + iy) - R_0(\lambda - iy)) f, f\rangle + \langle (R_0(\lambda + iy) - R_0(\lambda - iy)) g, g\rangle) \, d\lambda \\
&= \frac{1}{2} (\|f\|_2^2 + \|g\|_2^2).
\end{aligned} \tag{3.27}$$

The remaining terms are absolutely integrable due to smoothing estimates:

$$\int_{-\infty}^\infty \||V|^{1/2} R_0(\lambda \pm iy) f\|_2^2 \, d\lambda \leq C \|f\|_2^2. \tag{3.28}$$

By the Fourier inversion formula, (3.25) implies

$$\frac{i}{2\pi} \int_{\mathbb{R}} \chi(\lambda) \langle (R_V(\lambda + iy) - R_V(\lambda - iy)) f, g\rangle \, d\lambda = \langle f, g\rangle. \tag{3.29}$$

We then shift the integration contour toward the essential spectrum $(-\infty, -\mu] \cup [\mu, \infty)$, leaving behind circular contours around the finitely many (by Fredholm's Theorem) eigenvalues. Each contour integral becomes a corresponding Riesz projection.

What is left is P_c, the projection on the continuous spectrum. The integral is still absolutely convergent due to (3.26), (3.27), and smoothing estimates. (3.21) follows. $\square$

Using this representation, we proceed to derive $\dot H^{1/2}$ Strichartz estimates in the time-independent case.

Theorem 3.6. *Let Z be a solution of the linear Schrödinger equation*

$$i\partial_t Z + \mathcal{H} Z = F,\ Z(0) \text{ given}. \tag{3.30}$$

Assume that $\mathcal{H} = \mathcal{H}_0 + V$, $\langle x\rangle^m \partial^n V \in L^\infty$ for all m, $n \leq N$ and sufficiently large N, that V is matrix-valued as in (3.2), and that no eigenvalues or resonances are present in $(-\infty, -\mu] \cup [\mu, \infty)$. Then

$$\|P_c Z\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2, 6}_x} \leq C\Big(\|Z(0)\|_{\dot H^{1/2}} + \|F\|_{L^1_t \dot H^{1/2}_x + L^2_t \dot W^{1/2, 6/5}_x}\Big). \tag{3.31}$$

Proof. Let $F \in L^1_t(L^1_x \cap L^2_x)$, $G \in L^\infty_t(L^1_x \cap L^2_x)$ have compact support in t and consider the forward time evolution

$$(TF)(t) = \int_{t>s} e^{i(t-s)\mathcal{H}} P_c F(s)\,ds. \tag{3.32}$$

$TF(t)$ is in L^2_x for all t and grows at most exponentially, so its Fourier transform is well-defined for $\operatorname{Im}\lambda < -\|V\|_\infty$ (where, in particular, $\|R_V(\lambda)\|_{2\to 2}$ is bounded):

$$\widehat{TF}(\lambda) = i R_V(\lambda) P_c \widehat F(\lambda). \tag{3.33}$$

Using the representation formula (3.21) for P_c given by Lemma 3.5, for f, $g \in L^2$

$$\begin{aligned}\langle R_V(\lambda_0) P_c f, g\rangle &= \frac{1}{2\pi i} \int_{(-\infty, -\mu] \cup [\mu, \infty)} \big\langle R_V(\lambda_0)(R_V(\lambda - i0) - R_V(\lambda + i0)) f, g\big\rangle \,d\lambda \\ &= \frac{1}{2\pi i} \int_{(-\infty, -\mu] \cup [\mu, \infty)} \Big\langle \frac{1}{\lambda - \lambda_0}(R_V(\lambda - i0) - R_V(\lambda + i0)) f, g\Big\rangle \,d\lambda.\end{aligned} \tag{3.34}$$

Here we used the resolvent identity: for all λ_1, λ_2 in the resolvent set,

$$R_V(\lambda_1) - R_V(\lambda_2) = (\lambda_1 - \lambda_2) R_V(\lambda_1) R_V(\lambda_2).$$

From the absolute convergence of the integral for f, $g \in L^2$, it immediately follows that

$$\|R_V(\lambda_0) P_c\|_{L^2 \to L^2} \leq \frac{C}{d(\lambda_0, (-\infty, -\mu] \cup [\mu, \infty))}. \tag{3.35}$$

The limiting absorption principle provides that

$$\sup_{\lambda \in (-\infty, -\mu] \cup [\mu, \infty)} \|R_V(\lambda \pm i0)\|_{L^{6/5} \to L^6} < \infty \tag{3.36}$$

and thus, for $6/5 \leq p \leq 2$, $\Big\langle (R_V(\lambda - i0) - R_V(\lambda + i0)) f, g\Big\rangle \in L^{2p/(5p-6)}$. Therefore, for $\lambda_0 \notin (-\infty, -\mu] \cup [\mu, \infty)$ and $6/5 < p \leq 2$,

$$\|R_V(\lambda_0) P_c\|_{L^p \to L^{p'}} < \infty. \tag{3.37}$$

Also note that

$$R_V P_c = R_0 P_c - R_0 V P_c R_0 + R_0 V P_c R_V V R_0 \tag{3.38}$$

and consequently, for $\lambda_0 \notin (-\infty, -\mu] \cup [\mu, \infty)$,

$$\|R_V(\lambda_0)P_c\|_{L^{6/5}\to L^6} \leq \|R_0(\lambda_0)\|_{L^{6/5}\to L^6} + \|R_0(\lambda_0)\|^2_{L^{6/5}\to L^6}\|V\|_{L^{3/2}} + \\ + \|R_0(\lambda_0)\|_{L^{6/5}\to L^\infty}\|R_0(\lambda_0)\|_{L^1\to L^6}\|V\|^2_{L^{3/2}}\|R_V(\lambda_0)P_c\|_{L^{3/2}\to L^3} < \infty. \tag{3.39}$$

Using the limiting absorption principle, it follows that

$$\sup_{\lambda\in\mathbb{C}} \|R_V(\lambda)P_c\|_{L^{6/5}\to L^6} < \infty. \tag{3.40}$$

For $y > \|V\|_\infty$, both $e^{-yt}(TF)(t)$ and $e^{yt}G(t)$ belong to $L^2_{t,x}$. Taking the Fourier transform in t, by Plancherel's theorem

$$\begin{aligned}\int_{\mathbb{R}} \langle (TF)(t), G(t)\rangle\, dt &= \frac{1}{2\pi}\int_{\mathbb{R}} \big\langle \big(e^{-yt}(TF)(t)\big)^\wedge, \big(e^{yt}G(t)\big)^\wedge\big\rangle\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} \big\langle R_V(\lambda - iy)P_c\widehat{F}(\lambda - iy), \widehat{G(-t)}(\lambda - iy)\big\rangle\, d\lambda.\end{aligned} \tag{3.41}$$

Here $\langle\cdot,\cdot\rangle$ is the real dot product.

The pairing makes sense because R_VP_c is analytic in the lower half-plane, as a family of bounded operators from $L^{6/5}$ to L^6. Furthermore, $\widehat{F}$ and $\widehat{G}$ are analytic due to the compactness of the supports and, for every $y \in \mathbb{R}$, $\widehat{F}(\lambda+iy) \in L^2_\lambda L^{6/5}_x$ and likewise for $\widehat{G}$.

We shift the integration line toward the real axis, obtaining

$$\int_{\mathbb{R}} \langle (TF)(t), G(t)\rangle\, dt = \frac{1}{2\pi i}\int_{\mathbb{R}} \big\langle R_V(\lambda - i0)P_c\widehat{F}(\lambda), \widehat{G(-t)}(\lambda)\big\rangle\, d\lambda. \tag{3.42}$$

This leads to

$$\begin{aligned}\int_{\mathbb{R}} \langle (TF)(t), G(t)\rangle\, dt &\leq C\sup_{\lambda\in\mathbb{R}} \|R_V(\lambda - i0)P_c\|_{\mathcal{L}(L^{6/5},L^6)}\|\widehat{F}\|_{L^2_\lambda L^{6/5}_x}\|\widehat{G}\|_{L^2_\lambda L^{6/5}_x} \\ &\leq C\|F\|_{L^2_t L^{6/5}_x}\|G\|_{L^2_t L^{6/5}_x}.\end{aligned} \tag{3.43}$$

Then, we remove our previous assumption that F and G should have compact support. This establishes the Strichartz estimate

$$\Big\| \int_{t>s} e^{i(t-s)\mathcal{H}}P_cF(s)\, ds\Big\|_{L^2_t L^6_x} \leq C\|F\|_{L^2_t L^{6/5}_x}. \tag{3.44}$$

Using Duhamel's formula we obtain all the other Strichartz estimates.

For $\dot{H}^{1/2}$ Strichartz estimates, we start again from (3.42) and use the fact that

$$\sup_{\lambda\in\mathbb{R}} \|R_V(\lambda - i0)P_c\|_{\mathcal{L}(\dot{W}^{1/2,6/5},\dot{W}^{1/2,6})} < \infty. \tag{3.45}$$

Its derivation is exactly analogous to that of (3.40). □

3.3. **The time-dependent case.** We turn to time-dependent equations. We prove a concrete result in the case of interest, while at the same time placing it within a more general framework.

Given parameters $A(t)$ and $v(t) = (v_1(t), v_2(t), v_3(t))$, consider the family of isometries

$$U(t) = e^{\int_0^t (2v(s)\nabla + iA(s)\sigma_3)\, ds}. \tag{3.46}$$

The rate of change of $U(t)$ is then controlled by the norm

$$\|A(t)\|_{L^\infty_t} + \|v(t)\|_{L^\infty_t}. \tag{3.47}$$

The linearized Schrödinger equation under study has the form

$$i\partial_t R(t) + (\mathcal{H}_0 + U(t)^{-1} V U(t)) R(t) = F(t),\ R(0) \text{ given.} \tag{3.48}$$

Observe that the Hamiltonian at time t is $U(t)^{-1}(\mathcal{H}_0 + V)U(t)$, since $\mathcal{H}$ and U commute, i.e. it is $\mathcal{H}_0 + V = \mathcal{H}$ conjugated by $U(t)$.

Let $Z(t) = U(t)R(t)$. We rewrite the equation in the new variable Z, obtaining

$$i\partial_t Z(t) - i\partial_t U(t) U(t)^{-1} Z(t) + \mathcal{H}_0 Z(t) + V Z(t) = U(t)F(t),\ Z(0) = R(0). \tag{3.49}$$

In the subsequent lemma we encapsulate all the properties of U that we actually use in our study of (3.48) and (3.49).

Lemma 3.7. *Let $U(t)$ be defined by (3.46).*

1. *$U(t)$ is a strongly continuous family of $\dot W^{s,p}$ isometries, for $s \in \mathbb{R}$, $1 \leq p < \infty$.*
2. *For every t and $\tau \geq 0$, $U(t)$ and $U(\tau)$ commute with $\mathcal{H}_0$ and each other.*
3. *For some N and all $0 \leq s \leq 1$, there exists $\epsilon(N) > 0$ such that*

$$\begin{aligned}\|\langle x\rangle^{-N}(U(t)U(\tau)^{-1}e^{i(t-\tau)\mathcal{H}_0} - e^{i(t-\tau)\mathcal{H}_0})\langle x\rangle^{-N}\|_{\mathcal{L}(L^2_\tau \dot H^s_x, L^2_t \dot H^s_x)} \leq \\ \leq C(\|A(t)\|_{L^\infty_t} + \|v(t)\|_{L^\infty_t})^{\epsilon(N)}.\end{aligned} \tag{3.50}$$

Proof. It is easy to verify the first two properties directly. In particular, the Laplace operator Δ commutes with translations.

Concerning the last property, we compare

$$T(t,s) = e^{i(t-s)\mathcal{H}_0} \tag{3.51}$$

and

$$\tilde T(t,s) = e^{i(t-s)\mathcal{H}_0} e^{\int_s^t (2v(\tau)\nabla + iA(\tau)\sigma_3)\, d\tau}. \tag{3.52}$$

On one hand, from dispersive estimates,

$$\|T(t,s) - \tilde T(t,s)\|_{1\to\infty} \leq C|t-s|^{-3/2}. \tag{3.53}$$

On the other hand,

$$\|T(t,s) - \tilde T(t,s)\|_{2\to 2} \leq C. \tag{3.54}$$

It follows that for $N > 1$

$$\|\langle x\rangle^{-N}(T - \tilde T)\langle x\rangle^{-N}\|_{\mathcal{L}(L^2_{t,x}, L^2_{t,x})} < C. \tag{3.55}$$

This holds with a constant independent of A and v.

Finally, we prove the last stated property. Assume A and v are small. We first consider the case when there is no translation movement due to v and we only have to handle oscillation, due to A. Denote this kernel by $\tilde T_{osc}$, that is

$$\tilde T_{osc}(t,s) = e^{i(t-s)\mathcal{H}_0} e^{\int_s^t (iA(\tau)\sigma_3)\, d\tau}. \tag{3.56}$$

One has

$$e^{ia} - 1 \leq C \min(1, a) \tag{3.57}$$

and thus

$$\|T(t,s) - \tilde T_{osc}(t,s)\|_{2\to 2} \leq C \min(1, \|A\|_\infty |t-s|) \leq C\|A\|_\infty^\epsilon |t-s|^\epsilon. \tag{3.58}$$

Therefore, for large enough N, it follows from (3.53) and from (3.58) that

$$\|\langle x\rangle^{-N}(T - \tilde T_{osc})\langle x\rangle^{-N}\|_{\mathcal{L}(L^2_{t,x}, L^2_{t,x})} \leq C\|A\|_\infty^{\epsilon/3}. \tag{3.59}$$

Next, we consider the case when v is not necessarily zero. Let $d(t) = \int_0^t v(\tau)\,d\tau$. Then

$$\begin{aligned} &e^{-i(t-s)\Delta} e^{(\int_s^t 2v(\tau)\,d\tau)\nabla} = \\ &= \frac{1}{(-4\pi i)^{3/2}} (t-s)^{-3/2} e^{i\left(\frac{|x-y|^2}{4(t-s)} - \frac{(x-y)(d(t)-d(s))}{t-s} + \frac{(d(t)-d(s))^2}{t-s}\right)}. \end{aligned} \tag{3.60}$$

We treat the last factor $e^{i\frac{(d(t)-d(s))^2}{t-s}}$ in the same manner in which we treated the factors containing A. Consider the kernel that contains those factors together with $e^{i\frac{(d(t)-d(s))^2}{t-s}}$, leaving aside $e^{\frac{(x-y)(d(t)-d(s))}{t-s}}$:

$$\tilde{T}_1(t,s) = e^{i(t-s)\mathcal{H}_0} e^{\int_s^t i(A(\tau)\sigma_3)\,d\tau} e^{i\frac{(d(t)-d(s))^2}{t-s}}. \tag{3.61}$$

For this kernel as well, one has

$$\|\langle x\rangle^{-N}(T - \tilde{T}_1)\langle x\rangle^{-N}\|_{\mathcal{L}(L^2_{t,x}, L^2_{t,x})} \leq C(\|A\|_\infty^{\epsilon/3} + \|v\|_\infty^{\epsilon/3}). \tag{3.62}$$

Considering the fact that

$$\left| e^{i\frac{(x-y)(d(t)-d(s))}{t-s}} - 1 \right| \leq C \min(1, \|v\|_\infty(|x|+|y|)) \leq C\|v\|_\infty^\epsilon (|x|+|y|)^\epsilon, \tag{3.63}$$

it follows that for large enough N

$$\|\langle x\rangle^{-N}(\tilde{T}(t,s) - \tilde{T}_1(t,s))\langle x\rangle^{-N}\|_{2\to 2} \leq C\|v\|_\infty^\epsilon |t-s|^{-3/2}. \tag{3.64}$$

We also have the trivial bound

$$\|\tilde{T}(t,s) - \tilde{T}_1(t,s)\|_{2\to 2} \leq C. \tag{3.65}$$

Therefore

$$\|\langle x\rangle^{-N}(\tilde{T} - \tilde{T}_1)\langle x\rangle^{-N}\|_{\mathcal{L}(L^2_{t,x}, L^2_{t,x})} \leq C\|v\|_\infty^{\epsilon/3}. \tag{3.66}$$

Overall, we find that

$$\|\langle x\rangle^{-N}(\tilde{T} - T)\langle x\rangle^{-N}\|_{\mathcal{L}(L^2_{t,x}, L^2_{t,x})} \leq C(\|A\|_\infty^{\epsilon/3} + \|v\|_\infty^{\epsilon/3}). \tag{3.67}$$

Finally, note that

$$\begin{aligned} \|\partial_k \langle x\rangle^{-N}(\tilde{T} - T)\langle x\rangle^{-N} F\|_{L^2_{t,x}} &\leq C\big(\|\langle x\rangle^{-N-1}(\tilde{T} - T)\langle x\rangle^{-N} F\|_{L^2_{t,x}} + \\ &\quad + \|\langle x\rangle^{-N}(\tilde{T} - T)\langle x\rangle^{-N-1} F\|_{L^2_{t,x}} + \\ &\quad + \|\langle x\rangle^{-N}(\tilde{T} - T)\langle x\rangle^{-N} \partial_k F\|_{L^2_{t,x}}\big) \\ &\leq C(\|A\|_\infty^{\epsilon/3} + \|v\|_\infty^{\epsilon/3})\|F\|_{L^2_t \dot{H}^1_x}, \end{aligned} \tag{3.68}$$

in view of the fact that $\dot{H}^1 \subset L^6$. Consequently, for large enough N,

$$\|\langle x\rangle^{-N}(\tilde{T} - T)\langle x\rangle^{-N} F\|_{L^2_t \dot{H}^1_x} \leq C(\|A\|_\infty^{\epsilon/3} + \|v\|_\infty^{\epsilon/3})\|F\|_{L^2_t \dot{H}^1_x}. \tag{3.69}$$

Interpolating between this and L^2, we obtain boundedness on $L^2_t \dot{H}^s_x$, $0 \leq s \leq 1$. $\square$

Before proceeding with the proof of the Strichartz estimates in the time-dependent case, we require the following technical lemma.

Lemma 3.8. *Consider $\mathcal{H} = \mathcal{H}_0 + V$ as in* (3.2) *such that $\langle x\rangle^n \partial^m V \in L^\infty$ for every m, $n \leq N$ and some sufficiently large N. Assume that $\mathcal{H}$ has no eigenvalues or resonances embedded in $\sigma(\mathcal{H}_0)$. Then there exists a decomposition*

$$V - P_p\mathcal{H} = F_1(V)F_2(V), \tag{3.70}$$

where $P_p = I - P_c$, such that $F_1(V) \in \mathcal{L}(L^2, \langle x\rangle^{-\tilde N} L^2)$, $F_2(V) \in \mathcal{L}(\langle x\rangle^{\tilde N} L^2, L^2)$. Moreover, it is the case that

$$F_1(V) \in \mathcal{L}(\dot H^{1/2}, \langle x\rangle^{-\tilde N} \dot H^{1/2}),\ F_2(V) \in \mathcal{L}(\langle x\rangle^{\tilde N} \dot H^{1/2}, \dot H^{1/2}). \tag{3.71}$$

Proof. Simply take for some large n

$$\begin{aligned} F_1(V) &= (V - P_p\mathcal{H})\langle x\rangle^n \\ F_2(V) &= \langle x\rangle^{-n}. \end{aligned} \tag{3.72}$$

This works because both the range of $P_p\mathcal{H}$ and that of its adjoint are spanned by Schwartz functions. □

Theorem 3.9. *Consider equation* (3.48)*, for $\mathcal{H} = \mathcal{H}_0 + V$ as in* (3.2)*:*

$$i\partial_t Z - iv(t)\nabla Z + A(t)\sigma_3 Z + \mathcal{H}Z = F,\ Z(0) \text{ given}, \tag{3.73}$$

$$\mathcal{H} = \begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix} + \begin{pmatrix} W_1 & W_2 \\ -W_2 & W_1 \end{pmatrix}. \tag{3.74}$$

Assume that $\langle x\rangle^n \partial^m V \in L^\infty$, for all m, $n \leq N$ and sufficiently large N, that $\|A\|_\infty$ and $\|v\|_\infty$ are sufficiently small (in a manner that depends on V), and that there are no eigenvalues or resonances of $\mathcal{H}$ in $(-\infty, -\mu] \cup [\mu, \infty)$. Then

$$\|P_c Z\|_{L^\infty_t L^2_x \cap L^2_t L^{6,2}_x} \leq C\Big(\|Z(0)\|_2 + \|F\|_{L^1_t L^2_x + L^2_t L^{6/5,2}_x}\Big) \tag{3.75}$$

and

$$\|P_c Z\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} \leq C\Big(\|Z(0)\|_{\dot H^{1/2}} + \|F\|_{L^1_t \dot H^{1/2}_x + L^2_t \dot W^{1/2,6/5}_x}\Big). \tag{3.76}$$

Proof. To begin with, the fact that V is regular and of rapid decay guarantees the existence of a solution R, albeit one that may grow exponentially.

As in the time-independent case, let

$$\tilde Z = P_c Z,\ \tilde F = P_c F - 2iv(t)[P_c, \nabla]\tilde Z + A(t)[P_c, \sigma_3]\tilde Z. \tag{3.77}$$

The equation becomes

$$i\partial_t \tilde Z - iv(t)\nabla \tilde Z + A(t)\sigma_3 \tilde Z + \mathcal{H}\tilde Z = \tilde F,\ \tilde Z(0) = P_c R(0) \text{ given}. \tag{3.78}$$

The commutation terms

$$2iv(t)[P_c, \nabla]\tilde Z,\ A(t)[P_c, \sigma_3]\tilde Z \tag{3.79}$$

are small in the dual Strichartz norm (for small $\|v\|_\infty$ and $\|A\|_\infty$) and thus can be controlled by Strichartz inequalities and a simple fixed point argument.

Lemma 3.8 then provides a decomposition

$$V - P_p\mathcal{H} = F_1(V)F_2(V), \tag{3.80}$$

where $F_1(V)$ and $F_2^*(V)$ are bounded from L^2 to $\langle x\rangle^{-N} L^2$.

Denote

$$\tilde T_{F_2(V), F_1(V)} F(t) = \int_{-\infty}^t F_2(V) P_c e^{i(t-s)\mathcal{H}_0} U(t) U(s)^{-1} F_1(V) F(s)\, ds, \tag{3.81}$$

respectively

$$\tilde{T}_{F_2(V),I}F(t) = \int_{-\infty}^{t} F_2(V)P_c e^{i(t-s)\mathcal{H}_0} U(t)U(s)^{-1} F(s)\,ds. \tag{3.82}$$

By Duhamel's formula,

$$F_2(V)\tilde{Z}(t) = i\tilde{T}_{F_2(V),F_1(V)}F_2(V)\tilde{Z}(t) + \tilde{T}_{F_2(V),I}(-i\tilde{F}(s) + \delta_{s=0}\tilde{Z}(0)). \tag{3.83}$$

We compare the time-dependent kernel $\tilde{T}_{F_2(V),F_1(V)}$ with the time-independent one

$$T_{F_2(V),F_1(V)}F(t) = \int_{-\infty}^{t} F_2(V)P_c e^{i(t-s)\mathcal{H}_0} F_1(V)F(s)\,ds. \tag{3.84}$$

By Lemma 3.7 we obtain that

$$\lim_{\substack{\|A\|_\infty \to 0\\ \|v\|_\infty \to 0}} \|T_{F_2(V),F_1(V)} - \tilde{T}_{F_2(V),F_1(V)}\|_{\mathcal{L}(L^2_{t,x}, L^2_{t,x})} = 0. \tag{3.85}$$

The operator $I - iT_{F_1(V),F_2(V)}$ is invertible in $\mathcal{L}(L^2_{t,x}, L^2_{t,x})$. Indeed, its inverse is nothing but

$$(I - iT_{F_1(V),F_2(V)})^{-1}F(t) = F(t) - i\int_{-\infty}^{t} F_2(V)P_c e^{i(t-s)\mathcal{H}} F_1(V)F(s)\,ds. \tag{3.86}$$

The right-hand side is in $\mathcal{L}(L^2_{t,x}, L^2_{t,x})$ due to the Strichartz estimates of Theorem 3.6. The Duhamel formula proves that it is the inverse of the left-hand side, as claimed.

It follows that, when $\|A\|_\infty$ and $\|v\|_\infty$ are small enough, $I - i\tilde{T}_{F_1(V),F_2(V)}$ is also invertible in $\mathcal{L}(L^2_{t,x}, L^2_{t,x})$. This immediately implies the desired Strichartz estimates, as

$$\tilde{Z} = \big(\tilde{T}_{I,I} + \tilde{T}_{F_1(V),I}(I - iT_{F_1(V),F_2(V)})^{-1}\tilde{T}_{I,F_2(V)}\big)(\delta_{t=0}\tilde{Z}(0) - i\tilde{F}). \tag{3.87}$$

In regard to the $\dot{H}^{1/2}$ case, we have the decomposition

$$V - P_p\tilde{\mathcal{H}} = F_1(V)F_2(V), \tag{3.88}$$

where

$$F_1(V) \in \mathcal{L}(\dot{H}^{1/2}, \langle x\rangle^{-\tilde{N}}\dot{H}^{1/2}),\ F_2(V) \in \mathcal{L}(\langle x\rangle^{\tilde{N}}\dot{H}^{1/2}, \dot{H}^{1/2}) \tag{3.89}$$

respectively. By Lemma 3.8 it follows in the same manner that

$$\lim_{\substack{\|A\|_\infty \to 0\\ \|v\|_\infty \to 0}} \|T_{F_2(V),F_1(V)} - \tilde{T}_{F_2(V),F_1(V)}\|_{\mathcal{L}(L^2_t\dot{H}^{1/2}_x, L^2_t\dot{H}^{1/2}_x)} = 0. \tag{3.90}$$

Therefore we can invert $\tilde{T}_{F_2(V),F_1(V)}$ in the operator algebra $\mathcal{L}(L^2_t\dot{H}^{1/2}_x, L^2_t\dot{H}^{1/2}_x)$ and the proof of $\dot{H}^{1/2}$ Strichartz estimates proceeds by (3.87). □

Appendix A. Spaces of functions

Our computations take place mostly in Lebesgue and Sobolev spaces of functions defined on $\mathbb{R}^{3+1}$. This corresponds to three spatial dimensions and one extra dimension that accounts for time.

To begin with, we only consider measurable complex-valued functions or ones that take values within a finite-dimensional Banach space. In general, dealing with more general Banach space-valued functions, one must distinguish between weak and strong measurability.

We denote the Lebesgue norm of f by by $\|f\|_p$. The Sobolev spaces of integral order $W^{n,p}$ are then defined by

$$\|f\|_{W^{n,p}} = \Big(\sum_{|\alpha| \leq n} \|\partial^\alpha f\|_p^p \Big)^{1/p} \tag{A.1}$$

for $1 \leq p < \infty$ and

$$\|f\|_{W^{n,\infty}} = \sup_{|\alpha| \leq n} \|\partial^\alpha f\|_\infty. \tag{A.2}$$

when $p = \infty$. In addition, we consider Sobolev spaces of fractional order, both homogenous and inhomogenous:

$$\|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_p, \text{ respectively } \|f\|_{\dot W^{s,p}} = \||\nabla| f\|_p. \tag{A.3}$$

Here $\langle \nabla \rangle^s$ and $|\nabla|^s$ denote Fourier multipliers — multiplication on the Fourier side by $\langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$ and $|\xi|^s$ respectively.

When $p = 2$, the alternate notation $H^s = W^{s,2}$ or $\dot H^s = \dot W^{s,2}$ is customary.

Strichartz estimates may involve mixed space-time norms of the form

$$\|f\|_{L^p_t \dot W^{s,q}_x} = \Big(\int_{-\infty}^{\infty} \|f(x,t)\|^p_{\dot W^{s,q}_x} \, dt \Big)^{1/p}. \tag{A.4}$$

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